THEOREM 9.4 Rules of Differentiation

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be differentiable vector functions and $u(t)$ a differentiable scalar function.

(*i*) $\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)$

(*ii*) $\frac{d}{dt}[u(t)\mathbf{r}_1(t)] = u(t)\mathbf{r}_1'(t) + u'(t)\mathbf{r}_1(t)$

(*iii*) $\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t)$

(*iv*) $\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1(t) \times \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \times \mathbf{r}_2(t)$.

Note of caution.

Since the cross product of two vectors is not commutative, the order in which $\mathbf{r}_1$ and $\mathbf{r}_2$ appear in part (*iv*) of Theorem 9.4 must be strictly observed.

Integrals of Vector Functions If f, g, and h are integrable, then the indefinite and definite integrals of a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are defined, respectively, by

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

The indefinite integral of $\mathbf{r}$ is another vector function $\mathbf{R} + \mathbf{c}$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$.

Example 8 Integral of a Vector Function

If $\mathbf{r}(t) = 6t^2\mathbf{i} + 4e^{-2t}\mathbf{j} + 8\cos 4t\mathbf{k}$

then
$$\int \mathbf{r}(t)\,dt = \left[\int 6t^2\,dt\right]\mathbf{i} + \left[\int 4e^{-2t}\,dt\right]\mathbf{j} + \left[\int 8\cos 4t\,dt\right]\mathbf{k}$$
$$= [2t^3 + c_1]\mathbf{i} + [-2e^{-2t} + c_2]\mathbf{j} + [2\sin 4t + c_3]\mathbf{k}$$
$$= 2t^3\mathbf{i} - 2e^{-2t}\mathbf{j} + 2\sin 4t\mathbf{k} + \mathbf{c},$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. ❑

Length of a Space Curve If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is a smooth function, then it can be shown that the **length** of the smooth curve traced by $\mathbf{r}$ is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt. \qquad (3)$$

Arc Length As a Parameter A curve in the plane or in space can be parametrized in terms of the arc length s.

Example 9 Example 1 Revisited

Consider the helix of Example 1. Since $\|\mathbf{r}'(t)\| = \sqrt{5}$, it follows from (3) that the length of the curve from $\mathbf{r}(0)$ to an arbitrary point $\mathbf{r}(t)$ is

$$s = \int_0^t \sqrt{5}\,du = \sqrt{5}t,$$

where we have used u as a dummy variable of integration. Using $t = s/\sqrt{5}$, we obtain a vector equation of the helix as a function of arc length:

$$\mathbf{r}(s) = 2\cos\frac{s}{\sqrt{5}}\mathbf{i} + 2\sin\frac{s}{\sqrt{5}}\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}. \qquad (4)$$

Parametric equations of the helix are then

$$f(s) = 2\cos\frac{s}{\sqrt{5}}, \quad g(s) = 2\sin\frac{s}{\sqrt{5}}, \quad h(s) = \frac{s}{\sqrt{5}}.$$ ❑

The derivative of a vector function $\mathbf{r}(t)$ with respect to the parameter t is a tangent vector to the curve traced by $\mathbf{r}$. However, if the curve is parametrized in terms of arc length s, then $\mathbf{r}'(s)$ *is a unit tangent vector.* To see this, let a curve be described by $\mathbf{r}(s)$, where s is arc length. From (3), the length of the curve from $\mathbf{r}(0)$ to $\mathbf{r}(s)$ is $s = \int_0^s \|\mathbf{r}'(u)\|\,du$. Differentiation of this last equation with respect to s then yields $\|\mathbf{r}'(s)\| = 1$.

EXERCISES 9.1

Answers to selected odd-numbered problems begin on page ANS-21.

In Problems 1–10, graph the curve traced by the given vector function.

1. $\mathbf{r}(t) = 2\sin t\mathbf{i} + 4\cos t\mathbf{j} + t\mathbf{k};\ t \geq 0$
2. $\mathbf{r}(t) = \cos t\mathbf{i} + t\mathbf{j} + \sin t\mathbf{k};\ t \geq 0$
3. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \cos t\mathbf{k};\ t \geq 0$
4. $\mathbf{r}(t) = 4\mathbf{i} + 2\cos t\mathbf{j} + 3\sin t\mathbf{k}$
5. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle$
6. $\mathbf{r}(t) = \cosh t\mathbf{i} + 3\sinh t\mathbf{j}$
7. $\mathbf{r}(t) = \langle\sqrt{2}\sin t, \sqrt{2}\sin t, 2\cos t\rangle;\ 0 \leq t \leq \pi/2$
8. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$
9. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$
10. $\mathbf{r}(t) = \langle t\cos t, t\sin t, t^2\rangle$

In Problems 11–14, find the vector function that describes the curve C of intersection between the given surfaces. Sketch the curve C. Use the indicated parameter.

11. $z = x^2 + y^2,\ y = x;\ x = t$
12. $x^2 + y^2 - z^2 = 1,\ y = 2x;\ x = t$
13. $x^2 + y^2 = 9,\ z = 9 - x^2;\ x = 3\cos t$
14. $z = x^2 + y^2,\ z = 1;\ x = \sin t$
15. Given that $\mathbf{r}(t) = \frac{\sin 2t}{t}\mathbf{i} + (t-2)^5\mathbf{j} + t\ln t\mathbf{k}$, find $\lim_{t\to 0^+}\mathbf{r}(t)$.
16. Given that $\lim_{t\to a}\mathbf{r}_1(t) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\lim_{t\to a}\mathbf{r}_2(t) = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, find:
    (a) $\lim_{t\to a}[-4\mathbf{r}_1(t) + 3\mathbf{r}_2(t)]$
    (b) $\lim_{t\to a}\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)$

In Problems 17–20, find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for the given vector function.

17. $\mathbf{r}(t) = \ln t\mathbf{i} + \mathbf{j},\ t > 0$
18. $\mathbf{r}(t) = \langle t\cos t - \sin t, t + \cos t\rangle$
19. $\mathbf{r}(t) = \langle te^{2t}, t^3, 4t^2 - t\rangle$
20. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + \tan^{-1}t\mathbf{k}$

In Problems 21–24, graph the curve C that is described by $\mathbf{r}$ and graph $\mathbf{r}'$ at the indicated value of t.

21. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 6\sin t\mathbf{j};\ t = \pi/6$
22. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j};\ t = -1$
23. $\mathbf{r}(t) = 2\mathbf{i} + t\mathbf{j} + \frac{4}{1+t^2}\mathbf{k};\ t = 1$
24. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k};\ t = \pi/4$

In Problems 25 and 26, find parametric equations of the tangent line to the given curve at the indicated value of t.

25. $x = t,\ y = \frac{1}{2}t^2,\ z = \frac{1}{3}t^3;\ t = 2$
26. $x = t^3 - t,\ y = \frac{6t}{t+1},\ z = (2t+1)^2;\ t = 1$

In Problems 27–32, find the indicated derivative. Assume that all vector functions are differentiable.

27. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)]$
28. $\frac{d}{dt}[\mathbf{r}(t) \cdot (t\mathbf{r}(t))]$
29. $\frac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))]$
30. $\frac{d}{dt}[\mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))]$
31. $\frac{d}{dt}\left[\mathbf{r}_1(2t) + \mathbf{r}_2\left(\frac{1}{t}\right)\right]$
32. $\frac{d}{dt}[t^3\mathbf{r}(t^2)]$

In Problems 33–36, evaluate the given integral.

33. $\int_{-1}^{2}(t\mathbf{i} + 3t^2\mathbf{j} + 4t^3\mathbf{k})\,dt$
34. $\int_0^4(\sqrt{2t+1}\mathbf{i} - \sqrt{t}\mathbf{j} + \sin\pi t\mathbf{k})\,dt$
35. $\int(te^t\mathbf{i} - e^{-2t}\mathbf{j} + te^{t^2}\mathbf{k})\,dt$

Example 7 Using Tree Diagrams

If $z = u^2v^3w^4$ and $u = t^2$, $v = 5t - 8$, $w = t^3 + t$, find dz/dt.

Solution In this case the tree diagram indicates that

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$
$$= 2uv^3w^4(2t) + 3u^2v^2w^4(5) + 4u^2v^3w^3(3t^2 + 1).$$

Alternative Solution Differentiate $z = t^4(5t - 8)^3(t^3 + t)^4$ by the Product Rule. □

Remarks

If $w = F(x, y, z)$ has continuous partial derivatives of any order, then analogous to (3), the mixed partial derivatives are equal:

$$F_{xyz} = F_{yzx} = F_{zyx}, \quad F_{xzy} = F_{yxz} = F_{zxy}$$

and so on.

EXERCISES 9.4

Answers to selected odd-numbered problems begin on page ANS-22.

In Problems 1–6, sketch some of the level curves associated with the given function.

1. $f(x, y) = x + 2y$
2. $f(x, y) = y^2 - x$
3. $f(x, y) = \sqrt{x^2 - y^2 - 1}$
4. $f(x, y) = \sqrt{36 - 4x^2 - 9y^2}$
5. $f(x, y) = e^{y - x^2}$
6. $f(x, y) = \tan^{-1}(y - x)$

In Problems 7–10, describe the level surfaces but do not graph.

7. $F(x, y, z) = \dfrac{x^2}{9} + \dfrac{z^2}{4}$
8. $F(x, y, z) = x^2 + y^2 + z^2$
9. $F(x, y, z) = x^2 + 3y^2 + 6z^2$
10. $F(x, y, z) = 4y - 2z + 1$
11. Graph some of the level surfaces associated with $F(x, y, z) = x^2 + y^2 - z^2$ for $c = 0$, $c > 0$, and $c < 0$.
12. Given that
$$F(x, y, z) = \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{9},$$
find the x-, y-, and z-intercepts of the level surface that passes through $(-4, 2, -3)$.

In Problems 13–32, find the first partial derivatives of the given function.

13. $z = x^2 - xy^2 + 4y^5$
14. $z = -x^3 + 6x^2y^3 + 5y^2$
15. $z = 5x^4y^3 - x^2y^6 + 6x^5 - 4y$
16. $z = \tan(x^3y^2)$
17. $z = \dfrac{4\sqrt{x}}{3y^2 + 1}$
18. $z = 4x^3 - 5x^2 + 8x$
19. $z = (x^3 - y^2)^{-1}$
20. $z = (-x^4 + 7y^2 + 3y)^6$
21. $z = \cos^2 5x + \sin^2 5y$
22. $z = e^{x^2 \tan^{-1} y^2}$
23. $f(x, y) = xe^{x^3y}$
24. $f(\theta, \phi) = \phi^2 \sin \dfrac{\theta}{\phi}$
25. $f(x, y) = \dfrac{3x - y}{x + 2y}$
26. $f(x, y) = \dfrac{xy}{(x^2 - y^2)^2}$
27. $g(u, v) = \ln(4u^2 + 5v^3)$
28. $h(r, s) = \dfrac{\sqrt{r}}{s} - \dfrac{\sqrt{s}}{r}$
29. $w = 2\sqrt{x}y - ye^{y/z}$
30. $w = xy \ln xz$
31. $F(u, v, x, t) = u^2w^2 - uv^3 + vw\cos(ut^2) + (2x^2t)^4$
32. $G(p, q, r, s) = (p^2q^3)^{r^4s^5}$

In Problems 33 and 34, verify that the given function satisfies **Laplace's equation**:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

33. $z = \ln(x^2 + y^2)$
34. $z = e^{x^2 - y^2} \cos 2xy$

In Problems 35 and 36 verify that the given function satisfies the **wave equation**:

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

35. $u = \cos at \sin x$
36. $u = \cos(x + at) + \sin(x - at)$
37. The molecular concentration $C(x, t)$ of a liquid is given by $C(x, t) = t^{-1/2} e^{-x^2/kt}$. Verify that this function satisfies the **diffusion equation**:
$$\frac{k}{4}\frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t}$$
38. The pressure P exerted by an enclosed ideal gas is given by $P = k(T/V)$, where k is a constant, T is temperature, and V is volume. Find:
   (a) the rate of change of P with respect to V,
   (b) the rate of change of V with respect to T, and
   (c) the rate of change of T with respect to P.

In Problems 39–48, use the Chain Rule to find the indicated partial derivatives.

39. $z = e^{uv^2}$; $u = x^3$, $v = x - y^2$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
40. $z = u^2 \cos 4v$; $u = x^2y^3$, $v = x^3 + y^3$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
41. $z = 4x - 5y^2$; $x = u^4 - 8v^3$, $y = (2u - v)^2$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$
42. $z = \dfrac{x - y}{x + y}$; $x = \dfrac{u}{v}$, $y = \dfrac{v^2}{u}$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$
43. $w = (u^2 + v^2)^{3/2}$; $u = e^{-t} \sin \theta$, $v = e^{-t} \cos \theta$; $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial \theta}$
44. $w = \tan^{-1}\sqrt{uv}$; $u = r^2 - s^2$, $v = r^2s^2$; $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$
45. $R = rs^2t^4$; $r = ue^{v^2}$, $s = ve^{-u^2}$, $t = e^{u^2v^2}$; $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial v}$
46. $Q = \ln(pqr)$; $p = t^2 \sin^{-1} x$, $q = \dfrac{x}{t^2}$, $r = \tan^{-1} \dfrac{x}{t}$; $\dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial t}$
47. $w = \sqrt{x^2 + y^2}$; $x = \ln(rs + tu)$, $y = \dfrac{t}{u}\cosh rs$; $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial u}$
48. $s = p^2 + q^2 - r^2 + 4t$; $p = \phi e^{3\theta}$, $q = \cos(\phi + \theta)$, $r = \phi\theta^2$, $t = 2\phi + 8\theta$; $\dfrac{\partial s}{\partial \phi}, \dfrac{\partial s}{\partial \theta}$

In Problems 49–52, use (8) to find the indicated derivative.

49. $z = \ln(u^2 + v^2)$; $u = t^2$, $v = t^{22}$; $\dfrac{dz}{dt}$
50. $z = u^3v - uv^4$; $u = e^{-5t}$, $v = \sec 5t$; $\dfrac{dz}{dt}$
51. $w = \cos(3u + 4v)$; $u = 2t + \dfrac{\pi}{2}$, $v = -t - \dfrac{\pi}{4}$; $\left.\dfrac{dw}{dt}\right|_{t=\pi}$
52. $w = e^{xy}$; $x = \dfrac{4}{2t + 1}$, $y = 3t + 5$; $\left.\dfrac{dw}{dt}\right|_{t=0}$
53. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that **Laplace's equation** $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ becomes
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$
54. Van der Waals' equation of state for the real gas CO_2 is
$$P = \frac{0.08T}{V - 0.0427} - \frac{3.6}{V^2}.$$
If dT/dt and dV/dt are rates at which the temperature and volume change, respectively, use the Chain Rule to find dP/dt.
55. The equation of state for a thermodynamic system is $F(P, V, T) = 0$, where P, V, and T are pressure, volume, and temperature, respectively. If the equation defines V as a function of P and T, and also defines T as a function of V and P, show that
$$\frac{\partial V}{\partial T} = -\frac{\dfrac{\partial F}{\partial T}}{\dfrac{\partial F}{\partial V}} = \frac{1}{\dfrac{\partial T}{\partial V}}.$$
56. The voltage across a conductor is increasing at a rate of 2 volts/min and the resistance is decreasing at a rate of 1 ohm/min. Use $I = E/R$ and the Chain Rule to find the rate at which the current passing through the conductor is changing when $R = 50$ ohms and $E = 60$ volts.
57. The length of the side labeled x of the triangle in Figure 9.25 increases at a rate of 0.3 cm/s, the side labeled y increases at a rate of 0.5 cm/s, and the included angle θ increases at a rate of 0.1 rad/s. Use the Chain Rule to find the rate at which the area of the triangle is changing at the instant $x = 10$ cm, $y = 8$ cm, and $\theta = \pi/6$.

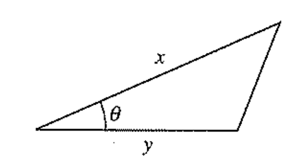


Figure 9.25 Triangle in Problem 57

58. A particle moves in 3-space so that its coordinates at any time are $x = 4\cos t$, $y = 4\sin t$, $z = 5t$, $t \ge 0$. Use the Chain Rule to find the rate at which its distance
$$w = \sqrt{x^2 + y^2 + z^2}$$
from the origin is changing at $t = 5\pi/2$ seconds.

Example 6 Max/Min of Directional Derivative

In Example 5 the maximum value of the directional derivative at F at $(1, -1, 2)$ is $\|\nabla F(1, -1, 2)\| = \sqrt{133}$. The minimum value of $D_{\mathbf{u}}F(1, -1, 2)$ is then $-\sqrt{133}$. □

Gradient Points in Direction of Most Rapid Increase of f Put yet another way, (10) and (11) state:

The gradient vector ∇f points in the direction in which f increases most rapidly, whereas $-\nabla f$ points in the direction of the most rapid decrease of f.

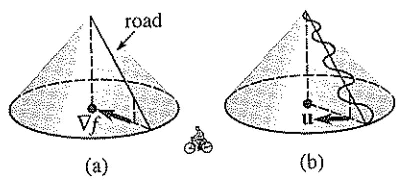


Figure 9.28 Model of a hill in Example 7

Example 7 Direction of Steepest Ascent

Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of $f(x, y) = 4 - \frac{2}{3}\sqrt{x^2 + y^2}$, $0 \le z \le 4$, shown in Figure 9.28(a) is a mathematical model of the hill. The gradient of f is

$$\nabla f(x, y) = \frac{2}{3}\left[\frac{-x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{-y}{\sqrt{x^2 + y^2}}\mathbf{j}\right] = \frac{2/3}{\sqrt{x^2 + y^2}}\mathbf{r},$$

where $\mathbf{r} = -x\mathbf{i} - y\mathbf{j}$ is a vector pointing to the center of the circular base.

Thus the steepest ascent up the hill is a straight road whose projection in the xy-plane is a radius of the circular base. Since $D_{\mathbf{u}}f = \text{comp}_{\mathbf{u}}\nabla f$, a bicyclist will zigzag, or seek a direction $\mathbf{u}$ other than ∇f, in order to reduce this component. □

Example 8 Direction to Cool Off Fastest

The temperature in a rectangular box is approximated by

$$T(x, y, z) = xyz(1 - x)(2 - y)(3 - z), \quad 0 \le x \le 1, \quad 0 \le y \le 2, \quad 0 \le z \le 3.$$

If a mosquito is located at $(\frac{1}{2}, 1, 1)$, in which direction should it fly to cool off as rapidly as possible?

Solution The gradient of T is

$$\nabla T(x, y, z) = yz(2 - y)(3 - z)(1 - 2x)\mathbf{i} + xz(1 - x)(3 - z)(2 - 2y)\mathbf{j} + xy(1 - x)(2 - y)(3 - 2z)\mathbf{k}.$$

Therefore, $\nabla T(\frac{1}{2}, 1, 1) = \frac{1}{4}\mathbf{k}$. To cool off most rapidly, the mosquito should fly in the direction of $-\frac{1}{4}\mathbf{k}$; that is, it should dive for the floor of the box, where the temperature is $T(x, y, 0) = 0$. □

EXERCISES 9.5

Answers to selected odd-numbered problems begin on page ANS-23.

In Problems 1–4, compute the gradient for the given function.

1. $f(x, y) = x^2 - x^3y^2 + y^4$
2. $f(x, y) = y - e^{-2x^2y}$
3. $F(x, y, z) = \dfrac{xy^2}{z^3}$
4. $F(x, y, z) = xy \cos yz$

In Problems 5–8, find the gradient of the given function at the indicated point.

5. $f(x, y) = x^2 - 4y^2$; $(2, 4)$
6. $f(x, y) = \sqrt{x^3y - y^4}$; $(3, 2)$
7. $F(x, y, z) = x^2z^2 \sin 4y$; $(-2, \pi/3, 1)$
8. $F(x, y, z) = \ln(x^2 + y^2 + z^2)$; $(-4, 3, 5)$

In Problems 9 and 10, use Definition 9.5 to find $D_{\mathbf{u}}f(x, y)$ given that $\mathbf{u}$ makes the indicated angle with the positive x-axis.

9. $f(x, y) = x^2 + y^2$; $\theta = 30°$
10. $f(x, y) = 3x - y^2$; $\theta = 45°$

In Problems 11–20, find the directional derivative of the given function at the given point in the indicated direction.

11. $f(x, y) = 5x^3y^6$; $(-1, 1)$, $\theta = \pi/6$
12. $f(x, y) = 4x + xy^2 - 5y$; $(3, -1)$, $\theta = \pi/4$
13. $f(x, y) = \tan^{-1}\dfrac{y}{x}$; $(2, -2)$, $\mathbf{i} - 3\mathbf{j}$
14. $f(x, y) = \dfrac{xy}{x + y}$; $(2, -1)$, $6\mathbf{i} + 8\mathbf{j}$
15. $f(x, y) = (xy + 1)^2$; $(3, 2)$, in the direction of $(5, 3)$
16. $f(x, y) = x^2 \tan y$; $\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$, in the direction of the negative x-axis
17. $F(x, y, z) = x^2y^2(2z + 1)^2$; $(1, -1, 1)$, $\langle 0, 3, 3\rangle$
18. $F(x, y, z) = \dfrac{x^2 - y^2}{z^2}$; $(2, 4, -1)$, $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
19. $F(x, y, z) = \sqrt{x^2y + 2y^2z}$; $(-2, 2, 1)$, in the direction of the negative z-axis
20. $F(x, y, z) = 2x - y^2 + z^2$; $(4, -4, 2)$, in the direction of the origin

In Problems 21 and 22, consider the plane through the points P and Q that is perpendicular to the xy-plane. Find the slope of the tangent at the indicated point to the curve of intersection of this plane and the graph of the given function in the direction of Q.

21. $f(x, y) = (x - y)^2$; $P(4, 2)$, $Q(0, 1)$; $(4, 2, 4)$
22. $f(x, y) = x^3 - 5xy + y^2$; $P(1, 1)$, $Q(-1, 6)$; $(1, 1, -3)$

In Problems 23–26, find a vector that gives the direction in which the given function increases most rapidly at the indicated point. Find the maximum rate.

23. $f(x, y) = e^{2x} \sin y$; $(0, \pi/4)$
24. $f(x, y) = xye^{x-y}$; $(5, 5)$
25. $F(x, y, z) = x^2 + 4xz + 2yz^2$; $(1, 2, -1)$
26. $F(x, y, z) = xyz$; $(3, 1, -5)$

In Problems 27–30, find a vector that gives the direction in which the given function decreases most rapidly at the indicated point. Find the minimum rate.

27. $f(x, y) = \tan(x^2 + y^2)$; $(\sqrt{\pi/6}, \sqrt{\pi/6})$
28. $f(x, y) = x^3 - y^3$; $(2, -2)$
29. $F(x, y, z) = \sqrt{xze^y}$; $(16, 0, 9)$
30. $F(x, y, z) = \ln\dfrac{xy}{z}$; $\left(\dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{3}\right)$
31. Find the directional derivative(s) of $f(x, y) = x + y^2$ at $(3, 4)$ in the direction of a tangent vector to the graph of $2x^2 + y^2 = 9$ at $(2, 1)$.
32. If $f(x, y) = x^2 + xy + y^2 - x$, find all points where $D_{\mathbf{u}}f(x, y)$ in the direction of $\mathbf{u} = (1/\sqrt{2})(\mathbf{i} + \mathbf{j})$ is zero.
33. Suppose $\nabla f(a, b) = 4\mathbf{i} + 3\mathbf{j}$. Find a unit vector $\mathbf{u}$ so that:
    (a) $D_{\mathbf{u}}f(a, b) = 0$,
    (b) $D_{\mathbf{u}}f(a, b)$ is a maximum, and
    (c) $D_{\mathbf{u}}f(a, b)$ is a minimum.
34. Suppose $D_{\mathbf{u}}f(a, b) = 6$. What is the value of $D_{-\mathbf{u}}f(a, b)$?
35. (a) If $f(x, y) = x^3 - 3x^2y^2 + y^3$, find the directional derivative of f at a point (x, y) in the direction of $\mathbf{u} = (1/\sqrt{10})(3\mathbf{i} + \mathbf{j})$.
    (b) If $F(x, y) = D_{\mathbf{u}}f(x, y)$ of part (a), find $D_{\mathbf{u}}F(x, y)$.
36. Consider the gravitational potential
$$U(x,y) = \frac{-Gm}{\sqrt{x^2 + y^2}},$$
where G and m are constants. Show that U increases or decreases most rapidly along a line through the origin.
37. If $f(x, y) = x^3 - 12x + y^2 - 10y$, find all points at which $\|\nabla f\| = 0$.
38. Suppose
$$D_{\mathbf{u}}f(a, b) = 7, \quad D_{\mathbf{v}}f(a, b) = 3$$
$$\mathbf{u} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}, \ \mathbf{v} = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$
Find $\nabla f(a, b)$.
39. Consider the rectangular plate shown in Figure 9.29. The temperature at a point (x, y) on the plate is given by $T(x, y) = 5 + 2x^2 + y^2$. Determine the direction an insect should take, starting at $(4, 2)$, in order to cool off as rapidly as possible.

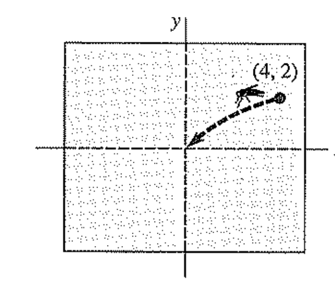


Figure 9.29 Insect in Problem 39

40. In Problem 39, observe that $(0, 0)$ is the coolest point of the plate. Find the path the cold-seeking insect, starting at $(4, 2)$, will take to the origin. If $\langle x(t), y(t)\rangle$ is the vector equation of the path, then use the fact that $-\nabla T(x, y) = \langle x'(t), y'(t)\rangle$. Why is this? [*Hint:* Remember separation of variables?]
41. The temperature at a point (x, y) on a rectangular metal plate is given by $T(x, y) = 100 - 2x^2 - y^2$. Find the path a heat-seeking particle will take, starting at $(3, 4)$, as it moves in the direction in which the temperature increases most rapidly.
42. The temperature T at a point (x, y, z) in space is inversely proportional to the square of the distance from (x, y, z) to the origin. It is known that $T(0, 0, 1) = 500$. Find the rate of change of T at $(2, 3, 3)$ in the direction of $(3, 1, 1)$. In which direction from $(2, 3, 3)$ does the temperature T increase most rapidly? At $(2, 3, 3)$ what is the maximum rate of change of T?

It follows from (5) that an equation of the tangent plane is

$$4(x-2) - 8(y-1) + 8(z-4) = 0 \quad \text{or} \quad x - 2y + 2z = 8.$$

Surfaces Given by $z = f(x, y)$ For a surface given explicitly by a differentiable function $z = f(x, y)$, we define $F(x, y, z) = f(x, y) - z$ or $F(x, y, z) = z - f(x, y)$. Thus, a point (x_0, y_0, z_0) is on the graph of $z = f(x, y)$ if and only if it is also on the level surface $F(x, y, z) = 0$. This follows from $F(x_0, y_0, z_0) = f(x_0, y_0) - z_0 = 0$.

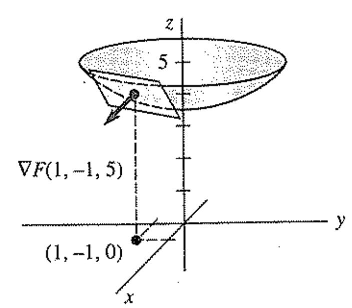


Figure 9.35 Tangent plane in Example 4

Example 4 Equation of Tangent Plane

Find an equation of the tangent plane to the graph of $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at $(1, -1, 5)$.

Solution Define $F(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4$ so that the level surface of F passing through the given point is $F(x, y, z) = F(1, -1, 5)$ or $F(x, y, z) = 0$. Now, $F_x = x$, $F_y = y$, and $F_z = -1$, so that

$$\nabla F(x, y, z) = x\mathbf{i} + y\mathbf{j} - \mathbf{k} \quad \text{and} \quad \nabla F(1, -1, 5) = \mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Thus, from (5) the desired equation is

$$(x - 1) - (y + 1) - (z - 5) = 0 \quad \text{or} \quad -x + y + z = 7.$$

See Figure 9.35. ☐

Normal Line Let $P(x_0, y_0, z_0)$ be a point on the graph of $F(x, y, z) = c$, where ∇F is not $\mathbf{0}$. The line containing $P(x_0, y_0, z_0)$ that is parallel to $\nabla F(x_0, y_0, z_0)$ is called the **normal line** to the surface at P. The normal line is perpendicular to the tangent plane to the surface at P.

Example 5 Normal Line to a Surface

Find parametric equations for the normal line to the surface in Example 4 at $(1, -1, 5)$.

Solution A direction vector for the normal line at $(1, -1, 5)$ is $\nabla F(1, -1, 5) = \mathbf{i} - \mathbf{j} - \mathbf{k}$. It follows that parametric equations for the normal line are $x = 1 + t$, $y = -1 - t$, $z = 5 - t$. ☐

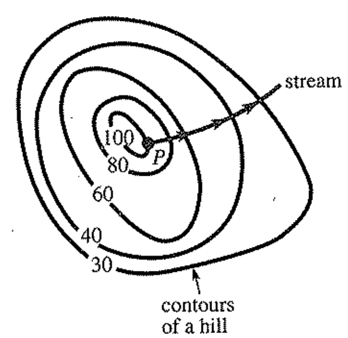


Figure 9.36 Stream is perpendicular to contours

Remarks

Water flowing down a hill chooses a path in the direction of the greatest change in altitude. Figure 9.36 shows the contours, or level curves, of a hill. As shown in the figure, a stream starting at point P will take a path that is perpendicular to the contours. After reading Sections 9.5 and 9.6, the student should be able to explain why.

EXERCISES 9.6

Answers to selected odd-numbered problems begin on page ANS-23.

In Problems 1–12, sketch the level curve or surface passing through the indicated point. Sketch the gradient at the point.

1. $f(x, y) = x - 2y$; $(6, 1)$
2. $f(x, y) = \dfrac{y + 2x}{x}$; $(1, 3)$
3. $f(x, y) = y - x^2$; $(2, 5)$
4. $f(x, y) = x^2 + y^2$; $(-1, 3)$
5. $f(x, y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$; $(-2, -3)$
6. $f(x, y) = \dfrac{y^2}{x}$; $(2, 2)$
7. $f(x, y) = (x - 1)^2 - y^2$; $(1, 1)$
8. $f(x, y) = \dfrac{y - 1}{\sin x}$; $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$
9. $F(x, y, z) = y + z$; $(3, 1, 1)$
10. $F(x, y, z) = x^2 + y^2 - z$; $(1, 1, 3)$
11. $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $(3, 4, 0)$
12. $F(x, y, z) = x^2 - y^2 + z$; $(0, -1, 1)$

In Problems 13 and 14, find the points on the given surface at which the gradient is parallel to the indicated vector.

13. $z = x^2 + y^2$; $4\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$
14. $x^3 + y^2 + z = 15$; $27\mathbf{i} + 8\mathbf{j} + \mathbf{k}$

In Problems 15–24, find an equation of the tangent plane to the graph of the given equation at the indicated point.

15. $x^2 + y^2 + z^2 = 9$; $(-2, 2, 1)$
16. $5x^2 - y^2 + 4z^2 = 8$; $(2, 4, 1)$
17. $x^2 - y^2 - 3z^2 = 5$; $(6, 2, 3)$
18. $xy + yz + zx = 7$; $(1, -3, -5)$
19. $z = 25 - x^2 - y^2$; $(3, -4, 0)$
20. $xz = 6$; $(2, 0, 3)$
21. $z = \cos(2x + y)$; $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}, -\dfrac{1}{\sqrt{2}}\right)$
22. $x^2y^3 + 6z = 10$; $(2, 1, 1)$
23. $z = \ln(x^2 + y^2)$; $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right)$
24. $z = 8e^{-2y}\sin 4x$; $\left(\dfrac{\pi}{24}, 0, 4\right)$

In Problems 25 and 26, find the points on the given surface at which the tangent plane is parallel to the indicated plane.

25. $x^2 + y^2 + z^2 = 7$; $2x + 4y + 6z = 1$
26. $x^2 - 2y^2 - 3z^2 = 33$; $8x + 4y + 6z = 5$
27. Find points on the surface $x^2 + 4x + y^2 + z^2 - 2z = 11$ at which the tangent plane is horizontal.
28. Find points on the surface $x^2 + 3y^2 + 4z^2 - 2xy = 16$ at which the tangent plane is parallel to (a) the xz-plane (b) the yz-plane, and (c) the xy-plane.

In Problems 29 and 30, show that the second equation is an equation of the tangent plane to the graph of the first equation at (x_0, y_0, z_0).

29. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} + \dfrac{zz_0}{c^2} = 1$
30. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} + \dfrac{zz_0}{c^2} = 1$
31. Show that every tangent plane to the graph of $z^2 = x^2 + y^2$ passes through the origin.
32. Show that the sum of the x-, y-, and z-intercepts of every tangent plane to the graph of $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$, $a > 0$, is the number a.

In Problems 33 and 34, find parametric equations for the normal line at the indicated point. In Problems 35 and 36, find symmetric equations for the normal line.

33. $x^2 + 2y^2 + z^2 = 4$; $(1, -1, 1)$
34. $z = 2x^2 - 4y^2$; $(3, -2, 2)$
35. $z = 4x^2 + 9y^2 + 1$; $(\frac{1}{2}, \frac{1}{3}, 3)$
36. $x^2 + y^2 - z^2 = 0$; $(3, 4, 5)$
37. Show that every normal line to the graph $x^2 + y^2 + z^2 = a^2$ passes through the origin.
38. Two surfaces are said to be **orthogonal** at a point P of intersection if their normal lines at P are orthogonal. Prove that the surfaces given by $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at P if and only if $F_xG_x + F_yG_y + F_zG_z = 0$.

In Problems 39 and 40, use the result of Problem 38 to show that the given surfaces are orthogonal at a point of intersection.

39. $x^2 + y^2 + z^2 = 25$; $-x^2 + y^2 + z^2 = 0$
40. $x^2 - y^2 + z^2 = 4$; $z = 1/xy^2$

9.7 Divergence and Curl

Introduction In Section 9.1 we introduced the concept of vector function of one variable. In this section we examine vector functions of two and three variables.

Vector Fields Vector functions of two and three variables,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

are also called **vector fields**. For example, the motion of a wind or a fluid can be described by means of a *velocity field* because a vector can be assigned at each point representing the velocity of a particle at the point. See Figures 9.37(a) and 9.37(b). The

By dividing the last expression by $\Delta x\,\Delta y\,\Delta z$, we get the outward flux of $\mathbf{F}$ per unit volume:

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

It is this combination of partial derivatives that is given a special name.

DEFINITION 9.8 Divergence

The **divergence** of a vector field $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is the scalar function

$$\text{div}\,\mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

Observe that div $\mathbf{F}$ can also be written in terms of the del operator as:

$$\text{div}\,\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}P(x,y,z)+\frac{\partial}{\partial y}Q(x,y,z)+\frac{\partial}{\partial z}R(x,y,z). \qquad (4)$$

Example 2 Curl and Divergence

If $\mathbf{F}=(x^2y^3-z^4)\mathbf{i}+4x^5y^2z\,\mathbf{j}-y^4z^6\mathbf{k}$, find curl $\mathbf{F}$ and div $\mathbf{F}$.

Solution From (1),

$$\text{curl}\,\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x^2y^3-z^4&4x^5y^2z&-y^4z^6\end{vmatrix}$$

$$=(-4y^3z^6-4x^5y^2)\mathbf{i}-4z^3\mathbf{j}+(20x^4y^2z-3x^2y^2)\mathbf{k}.$$

From (4),

$$\text{div}\,\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}(x^2y^3-z^4)+\frac{\partial}{\partial y}(4x^5y^2z)+\frac{\partial}{\partial z}(-y^4z^6)$$

$$=2xy^3+8x^5yz-6y^4z^5.$$ □

We ask you to prove the following two important properties. If f is a scalar function with continuous second partial derivatives, then

$$\text{curl}(\text{grad}\,f)=\nabla\times\nabla f=\mathbf{0}. \qquad (5)$$

Also, if $\mathbf{F}$ is a vector field having continuous second partial derivatives, then

$$\text{div}(\text{curl}\,\mathbf{F})=\nabla\cdot(\nabla\times\mathbf{F})=0. \qquad (6)$$

See Problems 29 and 30 in Exercises 9.7.

Physical Interpretations The word *curl* was introduced by Maxwell* in his studies of electromagnetic fields. However, the curl is easily understood in connection with the flow of fluids. If a paddle device, such as shown in Figure 9.41, is inserted in a flowing fluid, then the curl of the velocity field $\mathbf{F}$ is a measure of the tendency of the fluid to turn the device about its vertical axis w. If curl $\mathbf{F}=\mathbf{0}$, then the flow of the fluid is said to be irrotational, which means that it is free of vortices or whirlpools that would cause the paddle to rotate.* In Figure 9.42 the axis w of the paddle points straight out of the page.

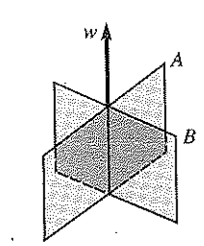
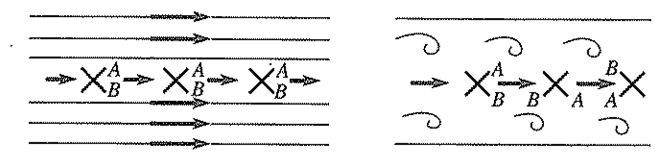


Figure 9.41 Paddle device

*James Clerk Maxwell (1831–1879), a Scottish physicist.

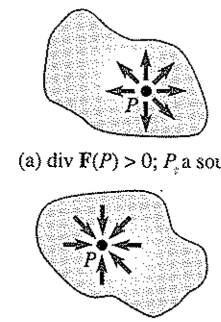


(a) Irrotational flow (b) Rotational flow

Figure 9.42 Irrotational flow in (a); Rotational flow in (b)

In the motivational discussion leading to Definition 9.8 we saw that the divergence of a velocity field $\mathbf{F}$ near a point $P(x, y, z)$ is the flux per unit volume. If div $\mathbf{F}(P)>0$, then P is said to be a **source** for $\mathbf{F}$, since there is a net outward flow of fluid near P; if div $\mathbf{F}(P)<0$, then P is said to be a **sink** for $\mathbf{F}$, since there is a net inward flow of fluid near P; if div $\mathbf{F}(P)=0$, there are no sources or sinks near P. See Figure 9.43.

The divergence of a vector field can also be interpreted as a measure of the rate of change of the density of the fluid at a point. In other words, div $\mathbf{F}$ is a measure of the fluid's compressibility. If $\nabla\cdot\mathbf{F}=0$, the fluid is said to be **incompressible**. In electromagnetic theory, if $\nabla\cdot\mathbf{F}=0$, the vector field $\mathbf{F}$ is said to be **solenoidal**.



(a) div $\mathbf{F}(P)>0$; P a source

(b) div $\mathbf{F}(P)<0$; P a sink

Figure 9.43 P a source in (a); P a sink in (b)

*In science texts the word *rotation* is sometimes used instead of *curl*. The symbol curl $\mathbf{F}$ is then replaced by rot $\mathbf{F}$.

EXERCISES 9.7

Answers to selected odd-numbered problems begin on page ANS-23.

In Problems 1–6, graph some representative vectors in the given vector field.

1. $\mathbf{F}(x,y)=x\mathbf{i}+y\mathbf{j}$
2. $\mathbf{F}(x,y)=-x\mathbf{i}+y\mathbf{j}$
3. $\mathbf{F}(x,y)=y\mathbf{i}+x\mathbf{j}$
4. $\mathbf{F}(x,y)=x\mathbf{i}+2y\mathbf{j}$
5. $\mathbf{F}(x,y)=y\mathbf{j}$
6. $\mathbf{F}(x,y)=x\mathbf{j}$

In Problems 7–16, find the curl and the divergence of the given vector field.

7. $\mathbf{F}(x,y,z)=xz\mathbf{i}+yz\,\mathbf{j}+xy\mathbf{k}$
8. $\mathbf{F}(x,y,z)=10yz\mathbf{i}+2x^2z\,\mathbf{j}+6x^3\mathbf{k}$
9. $\mathbf{F}(x,y,z)=4xy\mathbf{i}+(2x^2+2yz)\mathbf{j}+(3z^2+y^2)\mathbf{k}$
10. $\mathbf{F}(x,y,z)=(x-y)^3\mathbf{i}+e^{-yz}\mathbf{j}+xye^{2y}\mathbf{k}$
11. $\mathbf{F}(x,y,z)=3x^2y\mathbf{i}+2xz^3\mathbf{j}+y^4\mathbf{k}$
12. $\mathbf{F}(x,y,z)=5y^3\mathbf{i}+(\frac{1}{2}x^3y^2-xy)\mathbf{j}-(x^3yz-xz)\mathbf{k}$
13. $\mathbf{F}(x,y,z)=xe^{-z}\mathbf{i}+4yz^2\mathbf{j}+3ye^{-z}\mathbf{k}$
14. $\mathbf{F}(x,y,z)=yz\ln x\mathbf{i}+(2x-3yz)\mathbf{j}+xy^2z^3\mathbf{k}$
15. $\mathbf{F}(x,y,z)=xye^x\mathbf{i}-x^3yze^z\mathbf{j}+xy^2e^y\mathbf{k}$
16. $\mathbf{F}(x,y,z)=x^2\sin yz\mathbf{i}+z\cos xz^3\mathbf{j}+ye^{5xy}\mathbf{k}$

In Problems 17–24, let $\mathbf{a}$ be a constant vector and $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Verify the given identity.

17. div $\mathbf{r}=3$
18. curl $\mathbf{r}=\mathbf{0}$
19. $(\mathbf{a}\times\nabla)\times\mathbf{r}=-2\mathbf{a}$
20. $\nabla\times(\mathbf{a}\times\mathbf{r})=2\mathbf{a}$
21. $\nabla\cdot(\mathbf{a}\times\mathbf{r})=0$
22. $\mathbf{a}\times(\nabla\times\mathbf{r})=\mathbf{0}$
23. $\nabla\times[(\mathbf{r}\cdot\mathbf{r})\mathbf{a}]=2(\mathbf{r}\times\mathbf{a})$
24. $\nabla\cdot[(\mathbf{r}\cdot\mathbf{r})\mathbf{a}]=2(\mathbf{r}\cdot\mathbf{a})$

In Problems 25–32, verify the given identity. Assume continuity of all partial derivatives.

25. $\nabla\cdot(\mathbf{F}+\mathbf{G})=\nabla\cdot\mathbf{F}+\nabla\cdot\mathbf{G}$
26. $\nabla\times(\mathbf{F}+\mathbf{G})=\nabla\times\mathbf{F}+\nabla\times\mathbf{G}$
27. $\nabla\cdot(f\mathbf{F})=f(\nabla\cdot\mathbf{F})+\mathbf{F}\cdot\nabla f$
28. $\nabla\times(f\mathbf{F})=f(\nabla\times\mathbf{F})+(\nabla f)\times\mathbf{F}$
29. curl(grad f) $=\mathbf{0}$
30. div(curl $\mathbf{F}$) $=0$
31. div$(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot$ curl $\mathbf{F}-\mathbf{F}\cdot$ curl $\mathbf{G}$
32. curl(curl $\mathbf{F}$ + grad f) = curl(curl $\mathbf{F}$)
33. Show that

$$\nabla\cdot\nabla f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

This is known as the **Laplacian** and is also written $\nabla^2 f$.

34. Show that $\nabla\cdot(f\nabla f)=f\nabla^2 f+\|\nabla f\|^2$, where $\nabla^2 f$ is the Laplacian defined in Problem 33. [*Hint:* See Problem 27.]

35. Find curl(curl **F**) for the vector field $\mathbf{F} = xy\mathbf{i} + 4yz^2\mathbf{j} + 2xz\mathbf{k}$.

36. (a) Assuming continuity of all partial derivatives, show that curl(curl **F**) $= -\nabla^2\mathbf{F} +$ grad(div **F**), where

$$\nabla^2\mathbf{F} = \nabla^2(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \nabla^2P\mathbf{i} + \nabla^2Q\mathbf{j} + \nabla^2R\mathbf{k}$$

(b) Use the identity in part (a) to obtain the result in Problem 35.

37. Any scalar function f for which $\nabla^2 f = 0$ is said to be **harmonic**. Verify that $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic except at the origin. $\nabla^2 f = 0$ is called **Laplace's equation**.

38. Verify that

$$f(x, y) = \arctan\left(\frac{2}{x^2y^2 - 1}\right), \quad x^2 + y^2 \neq 1$$

satisfies Laplace's equation in two variables

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

39. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a mass m_1 and let the mass m_2 be located at the origin. If the force of gravitational attraction is

$$\mathbf{F} = -\frac{Gm_1m_2}{\|\mathbf{r}\|^3}\mathbf{r}$$

verify that curl $\mathbf{F} = \mathbf{0}$ and div $\mathbf{F} = 0$, $\mathbf{r} \neq \mathbf{0}$.

40. Suppose a body rotates with a constant angular velocity $\boldsymbol{\omega}$ about an axis. If **r** is the position vector of a point P on the body measured from the origin, then the linear velocity vector **v** of rotation is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. See Figure 9.44. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$, show that $\boldsymbol{\omega} = \frac{1}{2}$ curl **v**.

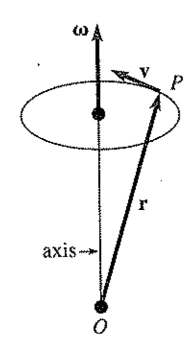


Figure 9.44 Rotating body in Problem 40

In Problems 41 and 42, assume that f and g have continuous second partial derivatives. Show that the given vector field is solenoidal. [*Hint:* See Problem 31.]

41. $\mathbf{F} = \nabla f \times \nabla g$

42. $\mathbf{F} = \nabla f \times (f\nabla g)$

43. The velocity vector field for the two-dimensional flow of an ideal fluid around a cylinder is given by

$$\mathbf{F}(x, y) = A\left[\left(1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right)\mathbf{i} - \frac{2xy}{(x^2 + y^2)^2}\mathbf{j}\right]$$

for some positive constant A. See Figure 9.45.

(a) Show that when the point (x, y) is far from the origin, $\mathbf{F}(x, y) \approx A\mathbf{i}$.

(b) Show that **F** is irrotational.

(c) Show that **F** is incompressible.

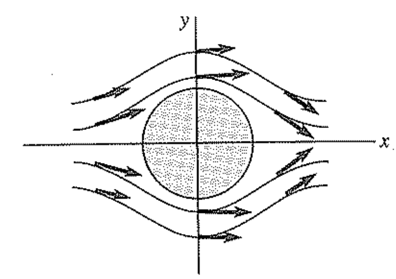


Figure 9.45 Vector field in Problem 43

44. If $\mathbf{E} = \mathbf{E}(x, y, z, t)$ and $\mathbf{H} = \mathbf{H}(x, y, z, t)$ represent electric and magnetic fields in empty space, then Maxwell's equations are

$$\text{div}\,\mathbf{E} = 0, \quad \text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\text{div}\,\mathbf{H} = 0, \quad \text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where c is the speed of light. Use the identity in Problem 36(a) to show that **E** and **H** satisfy

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \quad \nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}.$$

45. Consider the vector field $\mathbf{F} = x^2yz\mathbf{i} - xy^2z\mathbf{j} + (z + 5x)\mathbf{k}$. Explain why **F** is not the curl of another vector field **G**.

9.8 Line Integrals

Introduction The notion of the definite integral $\int_a^b f(x)\,dx$, that is, *integration of a function defined over an interval,* can be generalized to *integration of a function defined along a curve.* To this end we need to introduce some terminology about curves.

Terminology Suppose C is a curve parameterized by $x = f(t)$, $y = g(t)$, $a \le t \le b$, and A and B are the points $(f(a), g(a))$ and $(f(b), g(b))$, respectively. We say that:

(*i*) C is a **smooth curve** if f' and g' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(*ii*) C is **piecewise smooth** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end—that is, $C = C_1 \cup C_2 \cup \cdots \cup C_n$.

(*iii*) C is a **closed curve** if $A = B$.

(*iv*) C is a **simple closed curve** if $A = B$ and the curve does not cross itself.

(*v*) If C is not a closed curve, then the **positive direction** on C is the direction corresponding to increasing values of t.

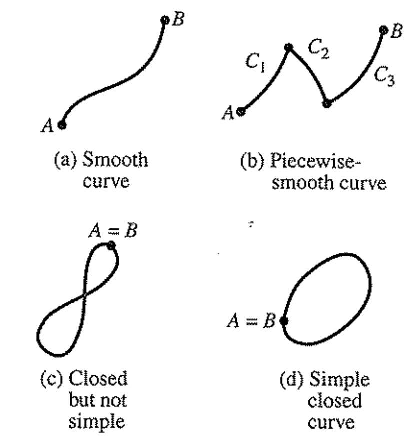


Figure 9.46 Various curves

Figure 9.46 illustrates each type of curve defined in (*i*)–(*iv*).

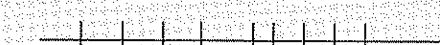
This same terminology carries over in a natural manner to curves in space. For example, a curve C defined by $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \le t \le b$, is smooth if f', g', and h' are continuous on $[a, b]$ and not simultaneously zero on (a, b).

Definite Integral Before defining integration along a curve, let us review the five steps leading to the definition of the definite integral.

$y = f(x)$

1. Let f be defined on a closed interval $[a, b]$.
2. Partition the interval $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$. Let P denote the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$



3. Let $\|P\|$ be the length of the longest subinterval. The number $\|P\|$ is called the **norm** of the partition P.
4. Choose a number x_k^* in each subinterval.

5. Form the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

The definite integral of a function of a single variable is given by the limit of a sum:

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Line Integrals in the Plane The following analogous five steps lead to the definitions of three **line integrals*** in the plane.

*An unfortunate choice of names. **Curve integrals** would be more appropriate.

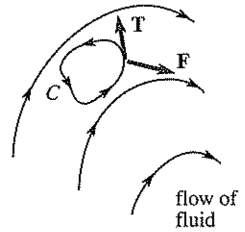
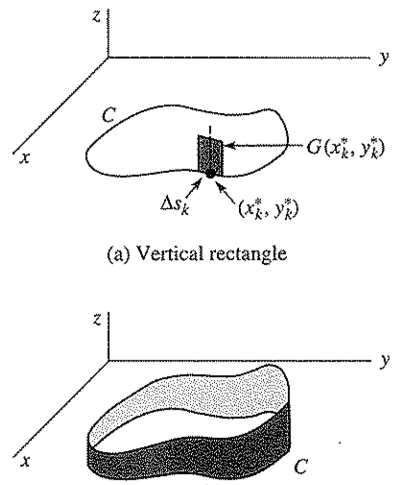


Figure 9.54 Does the velocity field turn the curve C?



(b) "Fence" or "curtain" of varying height $G(x, y)$ with base C

Figure 9.55 A geometric interpretation of a line integral

The units of work depend on the units of $\|\mathbf{F}\|$ and on the units of distance.

■ **Circulation** A line integral of a vector field **F** around a simple closed curve C is said to be the **circulation** of **F** around C; that is,

$$\text{circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

In particular, if **F** is the velocity field of a fluid, then the circulation is a measure of the amount by which the fluid tends to turn the curve C by rotating, or circulating, around it. For example, if **F** is perpendicular to **T** for every (x, y) on C, then $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$, and the curve does not move at all. On the other hand, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds > 0$ and $\int_C \mathbf{F} \cdot \mathbf{T}\, ds < 0$ mean that the fluid tends to rotate C in the counterclockwise and clockwise directions, respectively. See Figure 9.54.

Remarks

In the case of two variables, the line integral with respect to arc length $\int_C G(x, y)\, ds$ can be interpreted in a geometric manner when $G(x, y) \geq 0$ on C. In Definition 9.9 the symbol Δs_k represents the length of the kth subarc on the curve C. But from the figure accompanying that definition, we have the approximation $\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. With this interpretation of Δs_k we see from Figure 9.55(a) that the product $G(x_k^*, y_k^*)\,\Delta s_k$ is the area of a vertical rectangle of height $G(x_k^*, y_k^*)$ and width Δs_k. The integral $\int_C G(x, y)\, ds$ then represents the area of one side of a "fence" or "curtain" extending from the curve C in the xy-plane up to the graph of $G(x, y)$ that corresponds to points (x, y) on C. See Figure 9.55(b).

EXERCISES 9.8

Answers to selected odd-numbered problems begin on page ANS-24.

In Problems 1–4, evaluate $\int_C G(x, y)\, dx$, $\int_C G(x, y)\, dy$, and $\int_C G(x, y)\, ds$ on the indicated curve C.

1. $G(x, y) = 2xy$; $x = 5\cos t$, $y = 5\sin t$, $0 \leq t \leq \pi/4$
2. $G(x, y) = x^3 + 2xy^2 + 2x$; $x = 2t$, $y = t^2$, $0 \leq t \leq 1$
3. $G(x, y) = 3x^2 + 6y^2$; $y = 2x + 1$, $-1 \leq x \leq 0$
4. $G(x, y) = x^2/y^3$; $2y = 3x^{2/3}$, $1 \leq x \leq 8$

In Problems 5 and 6, evaluate $\int_C G(x, y, z)\, dx$, $\int_C G(x, y, z)\, dy$, $\int_C G(x, y, z)\, dz$, and $\int_C G(x, y, z)\, ds$ on the indicated curve C.

5. $G(x, y, z) = z$; $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq \pi/2$
6. $G(x, y, z) = 4xyz$; $x = \frac{1}{3}t^3$, $y = t^2$, $z = 2t$, $0 \leq t \leq 1$

In Problems 7–10, evaluate $\int_C (2x + y)\, dx + xy\, dy$ on the given curve C between $(-1, 2)$ and $(2, 5)$.

7. $y = x + 3$
8. $y = x^2 + 1$

9.

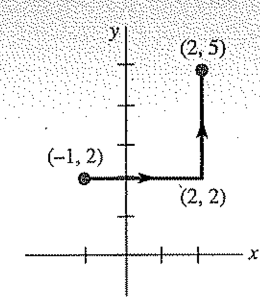


Figure 9.56 Curve C for Problem 9

10.

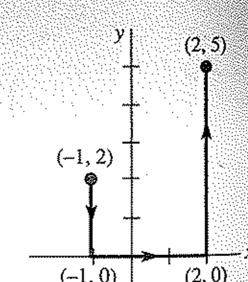


Figure 9.57 Curve C for Problem 10

In Problems 11–14, evaluate $\int_C y\, dx + x\, dy$ on the given curve C between $(0, 0)$ and $(1, 1)$.

11. $y = x^2$
12. $y = x$
13. C consists of the line segments from $(0, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 1)$.

14. C consists of the line segments from $(0, 0)$ to $(1, 0)$ and from $(1, 0)$ to $(1, 1)$.
15. Evaluate $\int_C (6x^2 + 2y^2)\, dx + 4xy\, dy$, where C is given by $x = \sqrt{t}$, $y = t$, $4 \leq t \leq 9$.
16. Evaluate $\int_C -y^2\, dx + xy\, dy$, where C is given by $x = 2t$, $y = t^3$, $0 \leq t \leq 2$.
17. Evaluate $\int_C 2x^3y\, dx + (3x + y)\, dy$, where C is given by $x = y^2$ from $(1, -1)$ to $(1, 1)$.
18. Evaluate $\int_C 4x\, dx + 2y\, dy$, where C is given by $x = y^3 + 1$ from $(0, -1)$ to $(9, 2)$.

In Problems 19 and 20, evaluate $\oint_C (x^2 + y^2)\, dx - 2xy\, dy$ on the given closed curve C.

19.

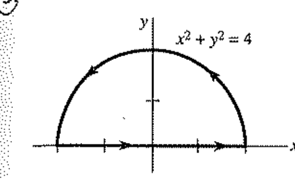


Figure 9.58 Closed curve C for Problem 19

20.

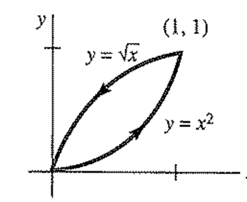


Figure 9.59 Closed curve C for Problem 20

In Problems 21 and 22, evaluate $\oint_C x^2y^3\, dx - xy^2\, dy$ on the given closed curve C.

21.

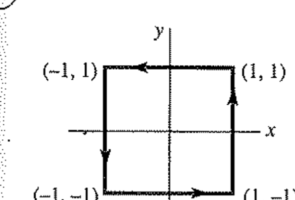


Figure 9.60 Closed curve C for Problem 21

22.

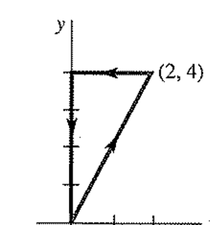


Figure 9.61 Closed curve C for Problem 22

23. Evaluate $\oint_C (x^2 - y^2)\, ds$, where C is given by

$$x = 5\cos t, \quad y = 5\sin t, \quad 0 \leq t \leq 2\pi$$

24. Evaluate $\int_{-C} y\, dx - x\, dy$, where C is given by

$$x = 2\cos t, \quad y = 3\sin t, \quad 0 \leq t \leq \pi$$

In Problems 25–28, evaluate $\int_C y\, dx + z\, dy + x\, dz$ on the given curve C between $(0, 0, 0)$ and $(6, 8, 5)$.

25. C consists of the line segments from $(0, 0, 0)$ to $(2, 3, 4)$ and from $(2, 3, 4)$ to $(6, 8, 5)$.
26. $x = 3t$, $y = t^3$, $z = \frac{5}{4}t^2$, $0 \leq t \leq 2$
27.

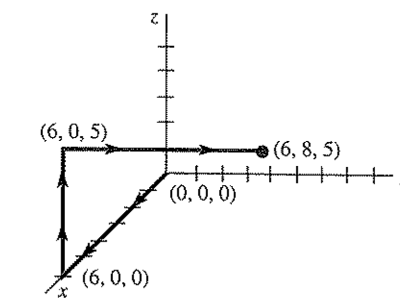


Figure 9.62 Closed curve C for Problem 27

28.

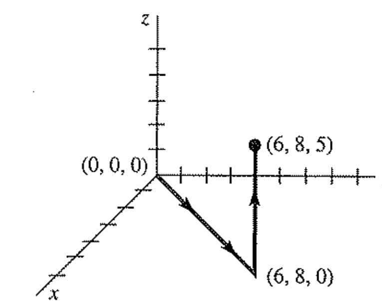


Figure 9.63 Closed curve C for Problem 28

In Problems 29 and 30, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

29. $\mathbf{F}(x, y) = y^3\mathbf{i} - x^2y\mathbf{j}$; $\mathbf{r}(t) = e^{-2t}\mathbf{i} + e^t\mathbf{j}$, $0 \leq t \leq \ln 2$
30. $\mathbf{F}(x, y, z) = e^x\mathbf{i} + xe^{xy}\mathbf{j} + xye^{xyz}\mathbf{k}$; $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$
31. Find the work done by the force $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ acting along $y = \ln x$ from $(1, 0)$ to $(e, 1)$.
32. Find the work done by the force $\mathbf{F}(x, y) = 2xy\mathbf{i} + 4y^2\mathbf{j}$ acting along the piecewise smooth curve consisting of the line segments from $(-2, 2)$ to $(0, 0)$ and from $(0, 0)$ to $(2, 3)$.
33. Find the work done by the force $\mathbf{F}(x, y) = (x + 2y)\mathbf{i} + (6y - 2x)\mathbf{j}$ acting counterclockwise once around the triangle with vertices $(1, 1)$, $(3, 1)$, and $(3, 2)$.
34. Find the work done by the force $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ acting along the curve given by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ from $t = 1$ to $t = 3$.
35. Find the work done by a constant force $\mathbf{F}(x, y) = a\mathbf{i} + b\mathbf{j}$ acting counterclockwise once around the circle $x^2 + y^2 = 9$.