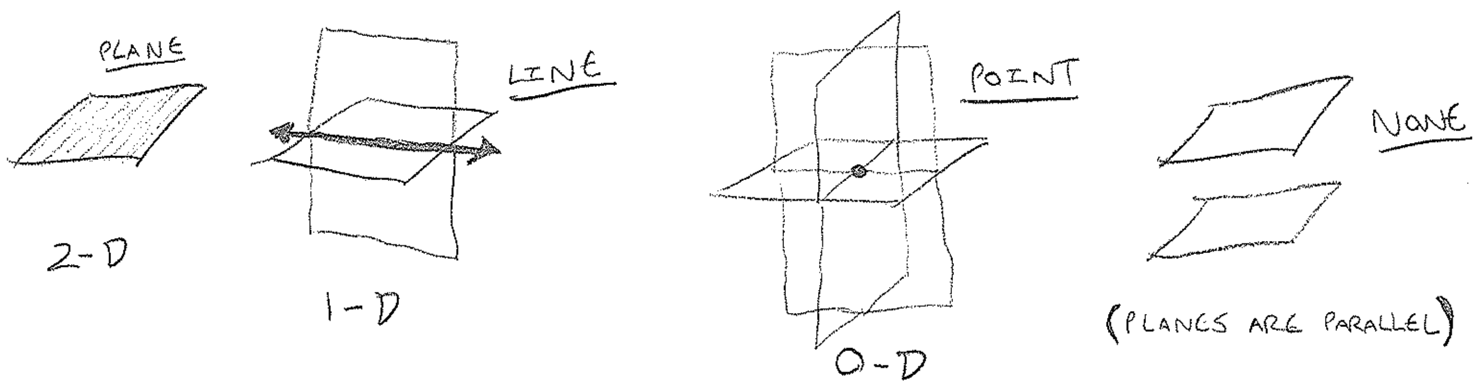
LINEAR ALGEBRA

SUPPOSE WE WANTED TO SOLVE A SYSTEM OF EQUATIONS

$$3x + y + z = 5$$
$$x + 3y - z = 3$$
$$-x - y + z = -1$$

THESE EQ'S ARE CALLED <u>LINEAR</u> SINCE THEY ONLY INVOLVE SCALAR MULTIPLES OF THE VARIABLES AND CONSTANTS. DO SOLUTIONS EXIST? IF SO, WHAT ARE THEY? WE CAN PICTURE OUR SOLUTION IN 3-SPACE AS THE INTERSECTION OF 3 PLANES. OUR SET OF SOLUTIONS TO THESE EQ'S CAN LOOK LIKE ONLY ONE OF A FEW THINGS:



EACH OF THESE SOLUTION SPACES HAS A DIMENSION WE CAN ASSOCIATE TO IT SINCE WE CAN PICTURE IT.

- HOW DO WE DEFINE THIS DIMENSION FOR MORE VARIABLES? (WE CAN'T PICTURE IT ANYMORE!)
- HOW CAN WE FIND THIS DIMENSION GIVEN JUST OUR EQ'S AND NOT HAVING TO DRAW PICTURES?

THIS IS THE START OF LINEAR ALGEBRA.

FIRST, SOME <u>NOTATION</u>.

$\mathbb{R}$ – REAL NUMBERS

$\mathbb{Z}$ – INTEGERS $\{0, \pm 1, \pm 2, \ldots\}$

$\mathbb{Q}$ – RATIONAL NUMBERS (FRACTIONS)

$\mathbb{C}$ – COMPLEX NUMBERS $\{a + ib \mid a, b \in \mathbb{R}\}$

THIS IS READ AS:

THE SET OF	ALL $a+ib$	SUCH THAT	a AND b ARE ELEMENTS OF THE REALS
$\{$	$a+ib$	$\mid$	$a,b \in \mathbb{R}$ $\}$

$\mathbb{R}^2$ – SET OF ALL POINTS (x, y) IN 2-SPACE

$\mathbb{R}^n$ – " $(x_1, x_2, \ldots, x_n)$ IN n-SPACE

s.t. – "SUCH THAT"

$A \Rightarrow B$ – STATEMENT A IMPLIES STATEMENT B

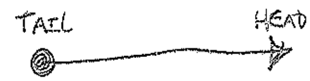
$A \Leftrightarrow B$ – STATEMENT A IFF STATEMENT B

IFF – IF AN ONLY IF

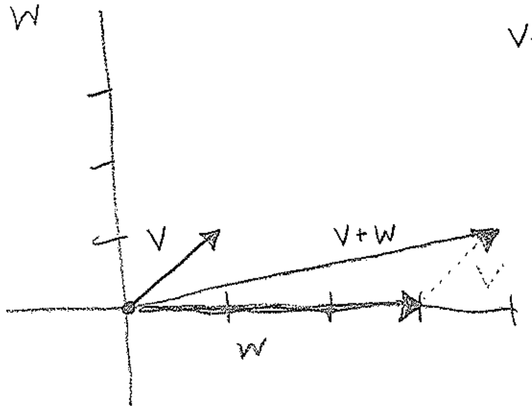
VECTORS IN $\mathbb{R}^2$

WE THINK OF VECTORS IN $\mathbb{R}^2$ AS PAIRS OF REAL NUMBERS (x,y) SUCH THAT WE ADD THEM COMPONENTWISE:

$$\underset{v}{(1,1)} + \underset{w}{(3,0)} = (1+3,\ 1+0) = \underset{v+w}{(4,1)}$$



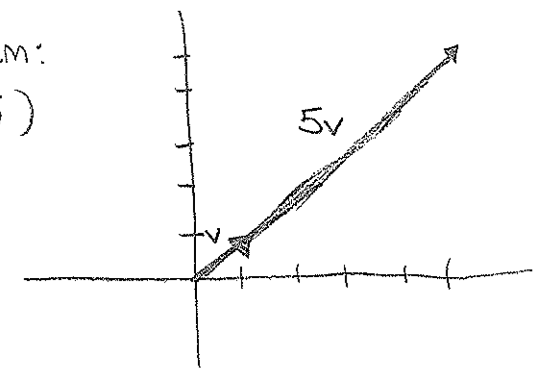
PICTORALLY:



WE CAN ADD V TO W BY MOVING THE TAIL OF V TO THE HEAD OF W.

WE CAN SCALE THEM:

$$5\underset{v}{(1,1)} = \underset{5v}{(5,5)}$$



WE CAN DO THE SAME THING FOR $\mathbb{R}^n$ w/ OPERATIONS:

$$(x_1,\dots,x_n)+(y_1,\dots,y_n) = (x_1+y_1,\ \dots,\ x_n+y_n)$$

$$c(x_1,\dots,x_n) = (cx_1,\dots,cx_n)$$

THE SPACES $\mathbb{R}^n$ THUS FORM A <u>VECTOR SPACE</u>. A VECTOR SPACE CONSISTS OF A SET OF OBJECTS $v \in \mathbb{V}$ SUCH THAT:

1. IF $v, w \in \mathbb{V}$, $v+w \in \mathbb{V}$ (WE CAN ADD THEM)
2. IF $v \in \mathbb{V}$, $cv \in \mathbb{V}$ FOR $c \in \mathbb{R}$ (WE CAN SCALE THEM)
3. THERE IS A ZERO VECTOR 0 SUCH THAT $0+v=v$ FOR ALL v.

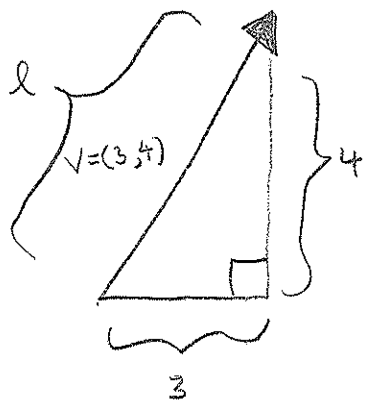
DOT PRODUCTS

FOR TWO VECTORS $V = (x_1, \dots, x_n)$ AND $W = (y_1, \dots, y_n) \in \mathbb{R}^n$, THE DOT PRODUCT IS:

$$V \cdot W = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

EX: $(1,2) \cdot (3,4) = (1)(3) + (2)(4) = 11$

THE LENGTH OF A VECTOR $(3,4)$ IS JUST $\sqrt{3^2+4^2} = 5$ BY PYTHAG:



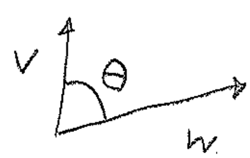
$$4^2 + 3^2 = \ell^2$$
$$25 = \ell^2$$
$$\underline{5 = \ell}$$

IN GENERAL, THE LENGTH OF A VECTOR $V = (x_1, \dots, x_n) \in \mathbb{R}^n$ IS DENOTED $\|V\|$ AND IS CALCULATED: $\|V\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

KEY PROPERTY OF $\cdot$

FOR TWO VECTORS V, W IN $\mathbb{R}^2$ OR $\mathbb{R}^3$,

$$V \cdot W = \|V\| \, \|W\| \cos\theta$$

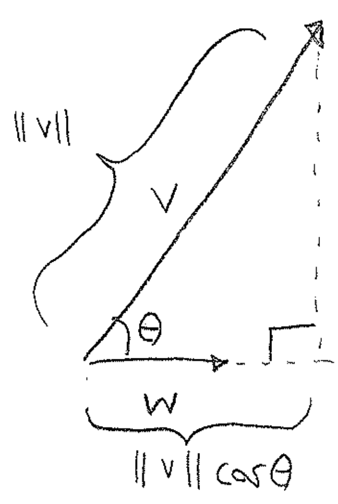


WHERE θ = ANGLE BETWEEN THEM

SO IF W IS A <u>UNIT</u> VECTOR (THIS MEANS $\|W\| = 1$) WE HAVE:

$$V \cdot W = \|V\| \cos\theta$$

BUT BY TRIG, THIS NUMBER HAS A GEOMETRIC INTERPRETATION:



THIS $V \cdot W$ IS THE LENGTH OF COMPONENT OF V IN THE DIRECTION OF W (CALLED THE PROJECTION ALONG W)

IN $\mathbb{R}^3$, WE OFTEN USE THE NOTATION

$(1,0,0) = i \qquad (0,1,0) = j \qquad (0,0,1) = k$ FOR SHORTHAND

IN $\mathbb{R}^n$, WE OFTEN USE

$(1,0,0,\dots,0) = e_1 \qquad (0,1,0,\dots,0) = e_2 \quad \dots \quad (0,0,\dots 0,1) = e_n$

<u>CROSS PRODUCTS</u> (ONLY IN $\mathbb{R}^3$)

SUPPOSE $v = (1,2,3)$ $w = (4,5,6)$

THEIR CROSS PRODUCT IS COMPUTED:

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} k$$

$$= (12-15)i - (6-12)j + (5-8)k$$

$$= -3i + 6j - 3k$$

$$= (-3, 6, -3)$$

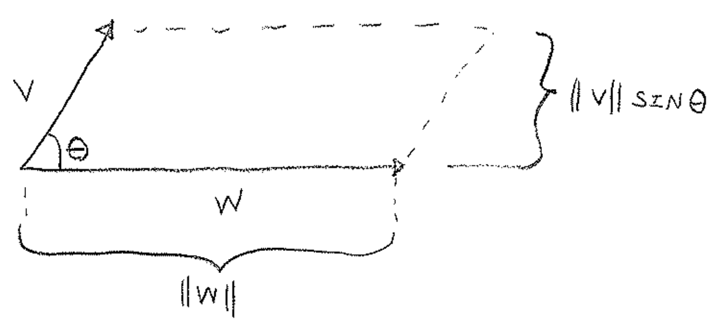
HERE $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ IS THE 2×2 DETERMINANT (WE'LL GET TO THIS LATER)

IT SATISFIES:

$\|v \times w\| = \|v\| \, \|w\| \sin\theta$ <u>AND</u> $v \times w$ IS PERPENDICULAR TO <u>BOTH</u> v AND w

GEOMETRICALLY:

IT IS THE AREA OF THE PARALLELOGRAM GIVEN BY THE VECTORS.



<u>NOTATION</u>: IN $\mathbb{R}^n$, WE WILL WRITE OUR VECTORS AS COLUMN VECTORS AS WELL. SO $(1,2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $(1,2,3) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ETC.

DEF: A LINEAR COMBINATION OF THE VECTORS $V_1, \dots, V_n$ IS ANY VECTOR OF THE FORM $C_1V_1 + C_2V_2 + \dots + C_nV_n = \sum_{i=1}^{n} C_iV_i$.

EX: A LINEAR COMBINATION (L.C. FROM NOW ON) OF $V = \begin{pmatrix}1\\0\end{pmatrix}$ AND $W = \begin{pmatrix}1\\1\end{pmatrix}$ IS $3\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}7\\4\end{pmatrix}$. IN FACT, EVERY VECTOR IN $\mathbb{R}^2$ IS A L.C. OF THESE TWO.

DEF: THE SPAN OF THE VECTORS $V_1, \dots, V_n$ IS DENOTED SPAN$\{V_1, \dots, V_n\}$, AND IS THE SET OF ALL LINEAR COMBINATIONS OF $V_1, \dots, V_n$. i.e.,

$$\text{SPAN}\{V_1, \dots, V_n\} = \{C_1V_1 + \dots + C_nV_n \mid C_1, \dots, C_n \in \mathbb{R}\}$$

NOTE: IS A VECTOR SPACE!!

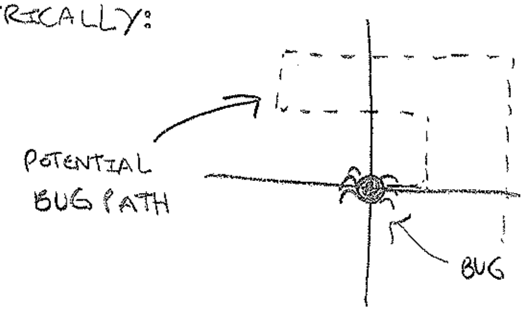
THINK OF SPAN$\{V_1, \dots, V_n\}$ AS THE POINTS IN $\mathbb{R}^n$ A BUG CAN GO TO STARTING AT THE ORIGIN!! IF IT CAN WALK ONLY IN THE DIRECTIONS OF $V_1, \dots,$ AND V_n.

EX'S: ① $\text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \left\{C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} \;\middle|\; C_1, C_2 \in \mathbb{R}\right\}$

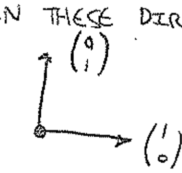
$= \left\{\begin{pmatrix}C_1\\C_2\end{pmatrix}\right\}$ ← WILL DROP THIS NOTATION

THUS WE CAN GET ANY VECTOR IN $\mathbb{R}^2$ BY CHOOSING C_1 AND C_2

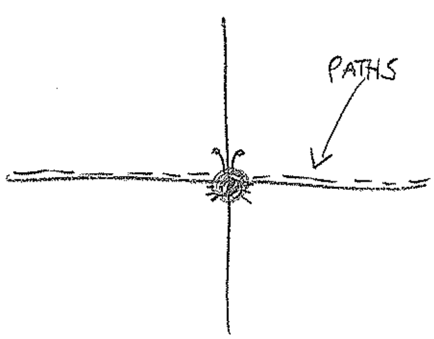
GEOMETRICALLY:



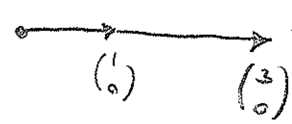
BUG CAN GO IN THESE DIRECTIONS:



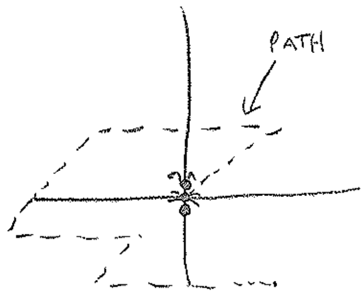
② $\text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\right\} = \left\{C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}3\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}C_1+3C_2\\0\end{pmatrix}\right\}$ = JUST THE X-AXIS



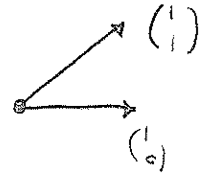
BUG CAN GO IN THESE DIRECTIONS: (B.C.G.I.T.D.)



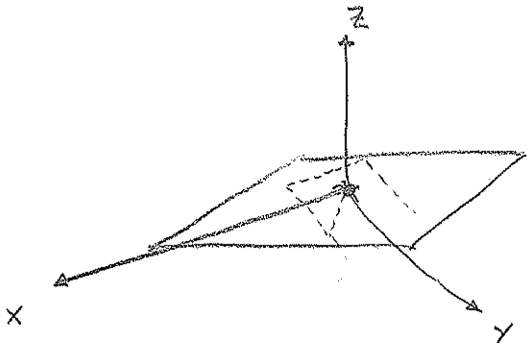
③ $\text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} = \left\{c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}\right\} = \left\{\begin{pmatrix}c_1+c_2\\c_2\end{pmatrix}\right\} = \mathbb{R}^2$



B.C.G.I.T.D.



④ $\text{SPAN}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = \left\{c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}c_1+c_3\\c_2+c_3\\0\end{pmatrix}\right\} = XY\text{-PLANE}$



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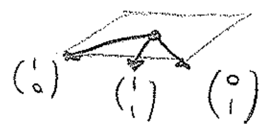

BUG IS <u>STUCK</u> ON XY-PLANE

NOTICE THAT IN EXAMPLES ② AND ④ WE COULD TOSS OUT 1 VECTOR AND STILL HAVE THE <u>SAME</u> SPAN.

IN ②, $\text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\right\} = \text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$

IN ④, $\text{SPAN}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = \text{SPAN}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

THIS IS PRECISELY BECAUSE THOSE VECTORS WE DROPPED WERE LINEAR COMBINATIONS OF THE OTHER VECTORS, i.e. THEY WERE NOT "INDEPENDENT" DIRECTIONS. THIS MOTIVATES:

DEF: VECTORS $V_1, \dots, V_n$ ARE SAID TO BE <u>LINEARLY INDEPENDENT</u> (L.I.) IF NO ONE OF THEM IS A LINEAR COMBINATION OF THE OTHERS. IF ONE <u>IS</u> A LINEAR COMBINATION OF THE OTHERS, THE VECTORS ARE SAID TO BE <u>LINEARLY DEPENDENT</u> (L.D.).

AN EQUIVALENT DEFINITION OF L.I. IS:

THE VECTORS $V_1, \ldots, V_n$ ARE L.I. IF THE ONLY LINEAR COMBINATION OF THEM THAT IS ZERO IS WHEN ALL THE CONSTANTS ARE ZERO. i.e.

$$c_1 V_1 + c_2 V_2 + \cdots + c_n V_n = 0 \underset{\text{(IMPLIES)}}{\Longrightarrow} c_1 = 0,\ c_2 = 0,\ \ldots,\ c_n = 0$$

NOW A SHORT PROOF TO SHOW THAT THESE DEFINITIONS ARE EQUIVALENT:

- SUPPOSE THAT NO ONE OF THE VECTORS $V_1, \ldots, V_n$ IS A LINEAR COMBINATION IN THE OTHERS. NOW IF $c_1 V_1 + c_2 V_2 + \cdots + c_n V_n = 0$, IF ANY OF THE C'S ARE NONZERO WE CAN DIVIDE BY IT (SAY $c_1 \neq 0$): $V_1 + \frac{c_2}{c_1} V_2 + \frac{c_3}{c_1} V_3 + \cdots + \frac{c_n}{c_1} V_n = 0$

  AND SO V_1 IS A LINEAR COMBINATION OF $V_2, \ldots, V_n$ (CONTRADICTION!)

  THUS, ALL THE C'S $= 0$. SO WE HAVE ONE IMPLICATION.
- SUPPOSE $c_1 V_1 + \cdots + c_n V_n = 0 \Longrightarrow c_1 = \cdots = c_n = 0$

  NOW IF ONE OF THE V'S (SAY V_1) WERE A LINEAR COMBINATION OF THE OTHERS, WE WOULD HAVE $V_1 = c_2 V_2 + \cdots + c_n V_n$

  SO $-V_1 + c_2 V_2 + \cdots + c_n V_n = 0$

  BUT THIS CANNOT HAPPEN BY OUR ASSUMPTION ABOVE!

  THUS, NO ONE OF THE V'S IS A LINEAR COMBINATION OF THE OTHERS.

DEF: THE DIMENSION OF A VECTOR SPACE V IS THE LARGEST NUMBER OF L.I. VECTORS ONE CAN HAVE IN IT.

EX: $\mathbb{R}^2$. $\begin{pmatrix}1\\0\end{pmatrix}$ AND $\begin{pmatrix}0\\1\end{pmatrix}$ ARE L.I. SINCE THEY ARE NOT SCALAR MULTIPLES OF ONE ANOTHER. NOW CAN WE ADD ANOTHER VECTOR $\begin{pmatrix}a\\b\end{pmatrix}$ s.t.

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}a\\b\end{pmatrix}\right\}$ ARE L.I.?

NO. $\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$ IS A LINEAR COMBINATION OF THE OTHERS.

THUS $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ ARE L.I. AND WE CANNOT ADD ANY MORE VECTORS TO GET A L.I. SET, SO $\mathrm{DIM}(\mathbb{R}^2) = 2$.

EX: $\mathbb{R}^3$. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ ARE L.I. SINCE IF FOR SOME c_1, c_2, c_3 WE HAVE:

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

THEN $\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow c_1 = c_2 = c_3 = 0$ (WE'RE USING THE OTHER EQUIV. DEFINITION OF L.I. HERE)

ALSO, WE CAN'T ADD ANOTHER VECTOR $\begin{pmatrix}a\\b\\c\end{pmatrix} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}$

TO GET A L.I. SET $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix}\right\}$ SINCE $\begin{pmatrix}a\\b\\c\end{pmatrix}$ IS A L.C. OF THE OTHERS

DEF: IN V IS AN n-DIMENSIONAL VECTOR SPACE, WE CALL A SET OF VECTORS $\{v_1, \ldots, v_n\}$ A <u>BASIS</u> IF THEY ARE L.I.

n OF THEM

SO A BASIS IS ONE OF THESE LARGEST SETS OF L.I. VECTORS.

EX: IN $\mathbb{R}^2$, $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ IS A BASIS

IN $\mathbb{R}^3$, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ IS A BASIS

IN $\mathbb{R}^n$, $\{e_1, \ldots, e_n\}$ IS A BASIS

THM: IF $V_1, \ldots, V_n$ IS A BASIS FOR V, THEN ANY VECTOR W IS A LINEAR COMBINATION OF THE $V_1, \ldots, V_n$.

PF: SINCE $\{V_1, \ldots, V_n\}$ IS A BASIS, IT IS THE LARGEST L.I. SET OF VECTORS AND THUS $\{V_1, \ldots, V_n, W\}$ ARE L.D.. THIS IMPLIES THAT THERE EXISTS CONSTANTS $c_1, \ldots, c_n, c_{n+1} \in \mathbb{R}$ NOT ALL ZERO s.t.

$$c_1 V_1 + c_2 V_2 + \cdots + c_n V_n + c_{n+1} W = 0$$

IF $c_{n+1} = 0$, WE HAVE: $c_1 V_1 + \ldots + c_n V_n = 0$

BUT THIS $\Longrightarrow$ $c_1 = \ldots = c_n = 0$ SINCE $\{V_1, \ldots, V_n\}$ ARE L.I.

SO $c_{n+1} \neq 0$, AND WE CAN DIVIDE BY IT:

$$\frac{c_1}{c_{n+1}} V_1 + \frac{c_2}{c_{n+1}} V_2 + \cdots + \frac{c_n}{c_{n+1}} V_n + W = 0$$

MOVE OVER HERE

SO W IS A LINEAR COMB. OF $V_1 \ldots V_n$. ∎

MATRICES

AN $m \times n$ MATRIX A IS AN ARRAY OF #'S WITH m ROWS AND n COLUMNS

WE TYPICALLY LABEL THE ELEMENTS a_{ij}

ROW COLUMN

A IS $m \times n$

ROW COLUMN

FOR EXAMPLE IF A IS 2×2:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(1) IF A AND B ARE BOTH $m \times n$, WE CAN ADD THEM:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

(2) WE CAN SCALE A MATRIX:

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

(3) IF A IS $m\times n$ AND B IS $n\times p$, WE CAN MULTIPLY TO GET AN $m\times p$ MATRIX

$$\underset{\boxed{m}\times n}{A}\ \underset{n\times\boxed{p}}{B} = \underset{\boxed{m}\times\boxed{p}}{C}$$

MUST BE EQUAL IN ORDER TO MULTIPLY!

IF $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

THEN $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

WHERE $c_{ij} = (i\text{TH ROW OF } A) \cdot (j\text{TH COLUMN OF } B)$

$\uparrow$ DOT PRODUCT

EX: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{array}{cc} \text{1ST ROW } A \cdot \text{1ST COL } B & \text{1ST ROW } A \cdot \text{2ND COL } B \\ \end{array}$

$$= \begin{pmatrix} (1,2)\cdot(0,-1) & (1,2)\cdot(1,2) \\ (3,4)\cdot(0,-1) & (3,4)\cdot(1,2) \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -4 & 11 \end{pmatrix}$$

- IF A IS $m\times n$ AND B IS $r\times s$, WITH $n \neq r$ WE CANNOT MULTIPLY

$$\underset{m\times n \neq r\times s}{AB}$$

- MATRIX MULTIPLICATION IS NOT COMMUTATIVE.

EX: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ THEN $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ A SWAPS X & Y

$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ THEN $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$ B DOUBLES X

$$AB\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 2x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 2x \end{pmatrix}$$

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 2y \\ x \end{pmatrix}$$

NOT THE SAME!

A FEW PROPERTIES OF MATRIX MULTIPLICATION / ADDITION:

(CAPS ARE MATRICES)

- $A + B = B + A$
- $(A+B)+C = A+(B+C)$
- $c_1(c_2A) = (c_1c_2)A \quad c_1 \text{ \& } c_2 \in \mathbb{R}$
- $c(A+B) = cA + cB \quad c \in \mathbb{R}$
- $(c_1+c_2)A = c_1A + c_2A \quad c_1, c_2 \in \mathbb{R}$
- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$

WE CAN VIEW AN $m \times n$ MATRIX AS A FUNCTION FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$

SINCE FOR A VECTOR $x \in \mathbb{R}^n$ (AN $n \times 1$ MATRIX)

$\underset{m \times n}{A}\,\underset{n \times 1}{x}$ IS <u>$m \times 1$</u>. SO WE TAKE A VECTOR $x \in \mathbb{R}^n$ TO $Ax \in \mathbb{R}^m$

WE WILL WANT TO THINK OF MATRICES AS FUNCTIONS A LOT!!

DEF: A MAP $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (NOTATION MEANS A FUNCTION TAKING VECTORS IN $\mathbb{R}^n$ TO VECTORS IN $\mathbb{R}^m$) IS CALLED A <u>LINEAR MAP</u> (OR FUNCTION) IF IT SATISFIES:

① $f(v+w) = f(v) + f(w)$

② $f(cv) = c\,f(v)$ FOR $c \in \mathbb{R}$

☆ FACT: EVERY LINEAR MAP FROM $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ IS GIVEN BY SOME $m \times n$ MATRIX, MEANING THAT IT TAKES A VECTOR v TO Av FOR SOME MATRIX A.

LET'S SEE WHY THIS MIGHT BE TRUE FOR MAPS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$.

FIRST OF ALL, MULTIPLICATION BY A MATRIX <u>DOES</u> DEFINE A LINEAR MAP SINCE:

$$A(v+w) = Av + Aw$$

$$A(cv) = c\,Av \quad \text{FOR } c \in \mathbb{R}$$

(FROM THE PROPERTIES OF MATRIX MULTIPLICATION)

NOW SUPPOSE $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ IS LINEAR, AND $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$, $f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$

THEN
$$f\begin{pmatrix}x\\y\end{pmatrix} = f\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= f\left(x\begin{pmatrix}1\\0\end{pmatrix}\right) + f\left(y\begin{pmatrix}0\\1\end{pmatrix}\right) \quad \text{BY ①}$$

$$= x\,f\begin{pmatrix}1\\0\end{pmatrix} + y\,f\begin{pmatrix}0\\1\end{pmatrix} \quad \text{BY ②}$$

$$= x\begin{pmatrix}a\\b\end{pmatrix} + y\begin{pmatrix}c\\d\end{pmatrix} \quad \text{BY ASSUMPTION}$$

$$= \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

SO f CORRESPONDS TO THE MATRIX WITH COLUMNS $f\begin{pmatrix}1\\0\end{pmatrix}, f\begin{pmatrix}0\\1\end{pmatrix}$.

IN GENERAL, FOR A LINEAR MAP $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

IT IS GIVEN BY A MATRIX WITH COLUMNS $\underbrace{f\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}}_{f(e_1)}, \underbrace{f\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}}_{f(e_2)}, \dots \underbrace{f\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}}_{f(e_n)}$

$$f\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} = \left(\begin{array}{c|c|c|c} f(e_1) & f(e_2) & \cdots & f(e_n) \end{array}\right)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

SO COLUMNS ARE WHERE f SENDS THE STANDARD BASIS VECTORS $e_1, \dots, e_n$ (IN THAT ORDER!)

NOW SUPPOSE f & g ARE LINEAR MAPS FROM $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ AND

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \qquad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$$

$$g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}r\\s\end{pmatrix} \qquad g\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}t\\u\end{pmatrix}$$

THEN $g(f(v+w)) = g(f(v)+f(w)) = g(f(v)) + g(f(w))$ ①

$g(f(cv)) = g(cf(v)) = c\, g(f(v))$ ②

SO THE COMPOSITION $g\circ f$ IS <u>LINEAR</u> AND CORRESPONDS TO A 2×2 MATRIX.

BY ABOVE, THIS MATRIX HAS COLUMNS $g\circ f\begin{pmatrix}1\\0\end{pmatrix}$ AND $g\circ f\begin{pmatrix}0\\1\end{pmatrix}$

$$g\circ f\begin{pmatrix}1\\0\end{pmatrix} = g\begin{pmatrix}a\\b\end{pmatrix} = g\left(a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$= g\left(a\begin{pmatrix}1\\0\end{pmatrix}\right) + g\left(b\begin{pmatrix}0\\1\end{pmatrix}\right)$ BY ①

$= a\, g\begin{pmatrix}1\\0\end{pmatrix} + b\, g\begin{pmatrix}0\\1\end{pmatrix}$ BY ②

$= a\begin{pmatrix}r\\s\end{pmatrix} + b\begin{pmatrix}t\\u\end{pmatrix}$ BY ASSUMPTION

$= \begin{pmatrix}ar+bt\\as+bu\end{pmatrix}$ COLUMN 1

SIMILARLY, $g\circ f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}cr+dt\\cs+du\end{pmatrix}$ COLUMN 2 THUS WE HAVE:

THE MATRIX CORRESPONDING TO THE COMPOSITION $g \circ f$ IS:

$$\underset{\substack{\uparrow \\ \text{MATRIX} \\ \text{FOR } g\circ f}}{\begin{pmatrix} ar+bt & cr+dt \\ as+bu & cs+du \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{MATRIX} \\ \text{FOR } g}}{\begin{pmatrix} r & t \\ s & u \end{pmatrix}} \underset{\substack{\uparrow \\ \text{MATRIX} \\ \text{FOR } f}}{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} \quad !!$$

<u>THUS</u> MATRIX MULTIPLICATION IS DEFINED THE WAY IT IS SO THAT WE HAVE THIS CORRESPONDENCE BETWEEN LINEAR MAPS AND MATRICES SUCH THAT COMPOSITION OF FUNCTIONS CORRESPONDS TO MATRIX MULTIPLICATION.

<u>NOW</u> THINGS WILL GET LESS THEORETICAL :)