LAST TIME: WE SOLVED DIFFY Q'S OF THE FORM

$$\star \quad y'' + P(x)y' + Q(x)y = 0$$

AND FOUND THAT WE COULD FIND SERIES SOLUTION CENTERED AT ANY POINT AT WHICH P AND Q WERE BOTH ANALYTIC. NOW WE FIND SOLUTIONS IN THE CASE OF P AND Q NOT BEING ANALYTIC AT A POINT.

SOLUTIONS ABOUT SINGULAR POINTS

WE CAN USE WHAT IS CALLED THE FROBENIUS THEOREM TO FIND SERIES SOLUTIONS AROUND SINGULAR POINTS OF THE DIFFY Q $\star$, AS LONG AS THE SINGULARITIES AREN'T TOO "BAD". LET'S BE MORE PRECISE:

DEF: A SINGULAR POINT $x = x_0$ IS SAID TO BE A REGULAR SINGULAR POINT IF THE FUNCTIONS $\tilde{P}(x) = (x - x_0)P(x)$, $\tilde{Q}(x) = (x - x_0)^2 Q(x)$ ARE BOTH ANALYTIC AT x_0. OTHERWISE, IT IS CALLED AN IRREGULAR SINGULAR POINT.

EX: $y'' + \frac{1}{x-1}y' + \frac{1}{(x-1)^2}y = 0$ HAS A REGULAR SINGULARITY AT $x = 1$:

$$P(x) = \frac{1}{x-1}, \quad \tilde{P}(x) = (x-1)P(x) = 1$$

$$Q(x) = \frac{1}{(x-1)^2}, \quad \tilde{Q}(x) = (x-1)^2 Q(x) = 1$$

BOTH ANALYTIC AT $x = 1$

EX: $y'' + \frac{1}{(x-1)^2}y' + \frac{1}{(x-1)^2}y = 0$ HAS AN IRREGULAR SINGULARITY AT $x = 1$:

$$\tilde{P} = (x-1)\frac{1}{(x-1)^2} = \frac{1}{x-1}$$ IS NOT ANALYTIC AT $x = 1$.

SO IN GENERAL A SINGULARITY (AT SAY $x = x_0$) IS REGULAR IFF THE MULTIPLICITY OF THE FACTOR $(x - x_0)$ IN THE DENOMINATOR OF $P(x)$ IS AT MOST 1, AND IN THE DENOMINATOR OF $Q(x)$ IS AT MOST 2.

EX: $$y'' + \frac{x-1}{(x+2)^2(x^2+1)}y' + \frac{x^2}{(x-1)(x+3)}y = 0$$

SINGULAR POINTS ARE $x = 1, -2, -3, \pm i$ AND ONLY $x = -2$ IS IRREGULAR

THM (FROBENIUS) IF $X = X_0$ IS A REGULAR SINGULAR POINT OF ☆, THEN

THERE EXISTS AT LEAST ONE SOLUTION OF THE FORM

$$Y = \sum_{n=0}^{\infty} c_n (X - X_0)^{n+r}$$

WHERE r IS SOME CONSTANT TO BE DETERMINED.

NOTE: IN GENERAL THIS MIGHT ONLY GET US ONE SOLUTION. TO FIND A SECOND L.I. SOLUTION WE CAN USE REDUCTION OF ORDER (SECT 3.2) WHICH WE HAVE NOT DONE. SO WE WILL LIMIT OURSELVES TO PROBLEMS IN WHICH THE ABOVE THM GIVES US TWO L.I. SOLUTIONS.

- WHAT WE WILL FIND IS THAT THE POSSIBLE VALUES FOR r (CALLED INDICIAL ROOTS) WILL BE THE ROOTS OF SOME 2ND DEGREE POLYNOMIAL, (SO WE GET TWO). SAY r_1 & r_2

THUS THERE ARE 3 CASES:

CASE 1 $r_1 \neq r_2$, $r_1 - r_2$ IS NOT AN INTEGER

THEN WE GET TWO L.I. SOLUTIONS

$$Y_1 = \sum c_n X^{n+r_1}, \quad Y_2 = \sum b_n X^{n+r_2}$$

CASE 2 $r_1 \neq r_2$, $r_1 - r_2$ IS AN INTEGER

WE MIGHT GET TWO SOLUTIONS OF THE FORM

CASE 3 $r_1 = r_2$, OUR SOLUTIONS TAKE THE FORM

$$Y_1 = \sum c_n X^{n+r_1}, \quad Y_2 = (\ln x) Y_1 + \sum b_n x^{n+r_1}$$

TO FIND THIS WE USE SECTION 3.2

SO LET'S DO AN EXAMPLE.

EX: $3xy'' + y' - y = 0$

$$3y'' + \frac{1}{x} y' - \frac{1}{x} y = 0$$

SO $X = 0$ IS A REGULAR SINGULAR POINT, WE CAN APPLY FROBENIUS:

$$Y = \sum_{n \geq 0} c_n X^{n+r} \quad \text{IS A SOLUTION FOR SOME } r$$

$$Y' = \sum_{n \geq 0} (n+r) c_n X^{n+r-1}$$

$$Y'' = \sum_{n \geq 0} (n+r)(n+r-1) c_n X^{n+r-2}$$

NOTICE THAT THESE SUMS ALL START AT $n = 0$ SINCE r COULD BE A FRACTION

SUBSTITUTE THIS BACK IN:

$$3x \sum_{n\geq 0}(n+r)(n+r-1)C_n x^{n+r-2} + \sum_{n\geq 0}(n+r)C_n x^{n+r-1} - \sum_{n\geq 0} C_n x^{n+r} = 0$$

FACTOR OUT x^r & MULTIPLY IN $3x$:

$$x^r\left[\underbrace{\sum_{n\geq 0}3(n+r)(n+r-1)C_n x^{n-1} + \sum_{n\geq 0}(n+r)C_n x^{n-1}}_{\text{COMBINE}} - \sum_{n\geq 0} C_n x^n\right] = 0$$

$$x^r\left[\underbrace{\sum_{n\geq 0}(n+r)\left[(3n+3r-3)+1\right]C_n x^{n-1}}_{\text{LET } k=n-1} - \underbrace{\sum_{n\geq 0} C_n x^n}_{\text{LET } k=n}\right] = 0$$

$$x^r\left[\sum_{k\geq -1}(k+r+1)\left(3(k+1)+3r-2\right)C_{k+1}x^k - \sum_{k\geq 0} C_k x^k\right] = 0$$

PULL OUT $k=-1$ TERM

$$x^r\left[r(3r-2)C_0 x^{-1} + \sum_{k\geq 0}(k+r+1)(3k+3r+1)C_{k+1}x^k - \sum_{k\geq 0} C_k x^k\right] = 0$$

NOW WE CAN COMBINE:

$$x^r\left[r(3r-2)C_0 x^{-1} + \sum_{k\geq 0}\left[(k+r+1)(3k+3r+1)C_{k+1} - C_k\right]x^k\right] = 0$$

SINCE THIS IS TRUE FOR ALL x WE CAN DIVIDE BY x^r AND DEDUCE AGAIN THAT EACH COEFFICIENT IN OUR SERIES IS ZERO:

$$r(3r-2)C_0 = 0 \quad \text{AND} \quad (k+r+1)(3k+3r+1)C_{k+1} - C_k = 0$$

SINCE WE DON'T WANT TO FORCE $C_0 = 0$, THE FIRST EQ. TELLS US $r=0$ OR $r=\frac{2}{3}$

THE ARE OUR INDICIAL ROOTS. FOR EACH r WE GET A RECURSION RELATION

$r_1 = 0$: $(k+1)(3k+1)C_{k+1} = C_k$

$$C_{k+1} = \frac{C_k}{(k+1)(3k+1)}$$

$k\geq 0$

$r_2 = \frac{2}{3}$: $\left(k+\frac{5}{3}\right)(3k+3)C_{k+1} = C_k$

$$(3k+5)(k+1)C_{k+1} = C_k$$

$$C_{k+1} = \frac{C_k}{(3k+5)(k+1)}$$

$k\geq 0$

SO FOR $r_1 = 0$:

$k=0$: $C_1 = C_0$

$k=1$: $C_2 = \frac{C_1}{2\cdot 4} = \frac{C_0}{2\cdot 4}$

$k=2$: $C_3 = \frac{C_2}{3\cdot 7} = \frac{C_0}{2\cdot 3\cdot 4\cdot 7}$

$k=3$: $C_4 = \frac{C_3}{4\cdot 10} = \frac{C_0}{2\cdot 3\cdot 4\cdot 4\cdot 7\cdot 10}$

$\vdots$

$$C_n = \frac{C_0}{n!(1\cdot 4\cdot 7\cdots\cdot(3n-2))}$$

AND $r_2 = \frac{2}{3}$:

$k=0$: $C_1 = \frac{C_0}{5}$

$k=1$: $C_2 = \frac{C_1}{8\cdot 2} = \frac{C_0}{2\cdot 5\cdot 8}$

$k=2$: $C_3 = \frac{C_2}{11\cdot 3} = \frac{C_0}{2\cdot 3\cdot 5\cdot 8\cdot 11}$

$k=3$: $C_4 = \frac{C_3}{14\cdot 4} = \frac{C_0}{2\cdot 3\cdot 4\cdot 5\cdot 8\cdot 11\cdot 14}$

$\vdots$

$$C_n = \frac{C_0}{n!\,(5\cdot 8\cdot 11\cdot\ldots\cdot(3n+2))}$$

THESE ARE SOLUTIONS FOR ANY C_0 SO LET $C_0 = 1$ AND WE GET TWO SOLUTIONS:

$$Y_1 = 1 + x + \frac{1}{2\cdot 4}x^2 + \frac{1}{2\cdot 3\cdot 4\cdot 7}x^3 + \frac{1}{2\cdot 3\cdot 4\cdot 4\cdot 7\cdot 10}x^4 + \ldots \quad (r_1 = 0)$$

$$Y_2 = x^{\frac{2}{3}}\left[1 + \frac{1}{5}x + \frac{1}{2\cdot 5\cdot 8}x^2 + \frac{1}{2\cdot 3\cdot 5\cdot 8\cdot 11}x^3 + \ldots\right] \quad \left(r_2 = \frac{2}{3}\right)$$

IF YOU WANT TO CHECK TO SEE IF YOUR INDICIAL ROOTS ARE CORRECT QUICKLY AND EASILLY (OR IF YOU ARE JUST ASKED TO FIND THE INDICIAL ROOTS) WE DISCUSS THE FOLLOWING:

FINDING r'S QUICKLY

SUPPOSE $x=0$ IS A REGULAR SINGULAR POINT OF $y'' + P(x)y' + Q(x)y = 0$.

THEN $\tilde{P}(x) = xP(x) = a_0 + a_1x + a_2x^2 + \ldots$ SINCE IT IS ANALYTIC (DEF OF REGULAR SINGULAR PT)

$\tilde{Q}(x) = x^2Q(x) = b_0 + b_1x + b_2x^2 + \ldots$

MULTIPLY OUR DIFFY Q BY x^2 AND SUBSTITUTE IN $y = \sum_{n\geq 0} C_n x^{n+r}$:

$$x^2y'' + x\underbrace{(xP(x))}_{\tilde{P}(x)}y' + \underbrace{(x^2Q(x))}_{\tilde{Q}(x)}y = 0$$

$$x^2\sum_{n\geq 0}(n+r-1)(n+r)C_nx^{n+r-2} + x(a_0 + a_1x + \ldots)\left(\sum_{n\geq 0}(n+r)C_nx^{n+r-1}\right) + (b_0 + b_1x + \ldots)\left(\sum_{n\geq 0}C_nx^{n+r}\right) = 0$$

NOW FACTOR OUT x^r, MULTIPLY IN OTHER x's.

$$x^r\left[\sum_{n\geq 0}(n+r-1)(n+r)c_n x^n + (a_0+a_1x+\dots)\left(\sum_{n\geq 0}(n+r)c_n x^n\right) + (b_0+b_1x+\dots)\left(\sum_{n\geq 0} c_n x^n\right)\right] = 0$$

NOW LOOK ONLY AT THE x^0 COEFFICIENT IN THE BRACKETED PART (THIS WILL $=0$):

$$(r-1)r\,c_0 + a_0 r\,c_0 + b_0 c_0 = 0$$

$$\big((r-1)r + a_0 r + b_0\big)c_0 = 0$$

SINCE WE WANT THIS TO BE TRUE FOR ANY c_0, WE WANT:

$$r(r-1) + a_0 r + b_0 = 0$$

TO FIND a_0 AND b_0, RECALL THAT $\tilde{P}(x) = xP(x) = a_0 + a_1x + \dots$

SO $\tilde{P}(0) = a_0$

SIMILARLY, $\tilde{Q}(x) = x^2Q(x) = b_0 + b_1x + \dots$

$\tilde{Q}(0) = b_0$

SO IN OUR PREVIOUS EXAMPLE:

$$3xy'' + y' - y = 0$$

$$y'' + \frac{1}{3x}y' - \frac{1}{3x}y = 0$$

$P(x) = \frac{1}{3x}$ $\qquad$ $Q(x) = -\frac{1}{3x}$

$\tilde{P}(x) = xP(x) = \frac{1}{3}$, $\tilde{P}(0) = \frac{1}{3} = a_0$ $\qquad$ $\tilde{Q}(x) = x^2Q(x) = -\frac{x}{3}$, $\tilde{Q}(0) = 0 = b_0$

SO OUR INDICIAL ROOTS ARE:

$$r(r-1) + \frac{1}{3}r = 0$$

$$3r(r-1) + r = 0$$

$$r(3r-3+1) = 0$$

$$r(3r-2) = 0$$

$$r = 0 \text{ OR } r = \frac{2}{3}$$

EX: $2xy'' + 5y' + xy = 0$

$P(x) = \frac{5}{2x}$, $Q(x) = \frac{1}{2}$ $x=0$ IS A REGULAR SINGULAR POINT, APPLY FROBENIUS:

$y = \sum_{n\geq 0} c_n x^{n+r}$, $y' = \sum_{n\geq 0} (n+r) c_n x^{n+r-1}$, $y'' = \sum_{n\geq 0} (n+r-1)(n+r) x^{n+r-2}$

SUBSTITUTE IN:

$$2x\sum_{n\geq 0}(n+r-1)(n+r)c_n x^{n+r-2} + 5\sum_{n\geq 0}(n+r)c_n x^{n+r-1} + x\sum_{n\geq 0} c_n x^{n+r} = 0$$

$$\sum_{n\geq 0} 2(n+r-1)(n+r)c_n x^{n+r-1} + \sum_{n\geq 0} 5(n+r)c_n x^{n+r-1} + \sum_{n\geq 0} c_n x^{n+r+1} = 0$$

FACTOR OUT x^r

$$x^r\left[\underbrace{\sum_{n\geq 0} 2(n+r-1)(n+r)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n\geq 0} 5(n+r)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n\geq 0} c_n x^{n+1}}_{k=n+1}\right] = 0$$

$$x^r\left[\sum_{k\geq -1} 2(k+r)(k+r+1)c_{k+1}x^k + \sum_{k\geq -1} 5(k+r+1)c_{k+1}x^k + \sum_{k\geq 1} c_{k-1}x^k\right] = 0$$

PULL OUT $k=-1,0$ TERMS & THEN COMBINE

WE'RE GOING TO OMIT THIS

$$2(r-1)rc_0x^{-1} + 2r(r+1)c_1 + 5rc_0x^{-1} + 5(r+1)c_1 + \ldots$$

$$\ldots + \sum_{k\geq 1}\left[2(k+r)(k+r+1)c_{k+1} + 5(k+r+1)c_{k+1} + c_{k-1}\right]x^k = 0$$

OUR x^{-1} ORDER TERMS $=0$:

$$[2(r-1)r + 5r]c_0 = 0$$

$$2r^2 + 3r = 0$$

$$r(2r+3) = 0$$

$$r_1 = 0,\ r_2 = -\frac{3}{2}$$

QUICK CHECK THAT OUR r's ARE CORRECT:

$\tilde{P}(x) = xP(x) = \frac{5}{2}$, $a_0 = \tilde{P}(0) = \frac{5}{2}$

$\tilde{Q}(x) = x^2Q(x) = \frac{1}{2}x^2$, $b_0 = \tilde{Q}(0) = 0$

SO OUR r's ARE

$$r(r-1) + a_0 r + b_0 = 0$$

$$r^2 - r + \frac{5}{2}r + 0 = 0$$

$$2r^2 - 2r + 5r = 0$$

$$2r^2 + 3r = 0$$

$$r = 0, -\frac{3}{2} \ ✓$$

NOW SINCE THESE r_1 & r_2 DO NOT DIFFER BY AN INTEGER ($r_1 - r_2 = \frac{3}{2}$ IS NOT AN INTEGER) EACH r WILL GET US A L.I. SOLUTION.

$r_1 = 0$:

LOOK AT THE x^0 TERM. (CONSTANT TERM):

$$\left[2r(r+1) + 5(r+1)\right]c_1 = 0 \quad \text{LET } r=0$$

$$5c_1 = 0$$

$$c_1 = 0$$

LOOK AT OUR RECURRENCE: $2k(k+1)c_{k+1} + 5(k+1)c_{k+1} + c_{k-1} = 0, \; k \geq 1$

$$(2k+5)(k+1)c_{k+1} = -c_{k-1}$$

$$c_{k+1} = \frac{-c_{k-1}}{(2k+5)(k+1)}$$

SO:

$$c_2 = \frac{-c_0}{2\cdot 7}$$

$$c_3 = \frac{-c_1}{\text{STUFF}} = 0 \quad (c_1 = 0)$$

$$c_4 = \frac{-c_2}{4\cdot 11} = \frac{c_0}{2\cdot 4\cdot 7\cdot 11}$$

$$c_5 = \frac{-c_3}{\text{STUFF}} = 0 \quad (c_3 = 0) \quad \text{(ALL ODD } c\text{'s} = 0)$$

$$y_1 = c_0\left[1 - \frac{1}{2\cdot 7}x^2 + \frac{1}{2\cdot 4\cdot 7\cdot 11}x^4 - \dots\right]$$

(c_0 CAN BE ANY NONZERO #)
IN PARTICULAR $c_0 = 1$ IS OK

$r_2 = -\frac{3}{2}$: LOOK AT THE CONSTANT TERM: $\left[2r(r+1) + 5(r+1)\right]c_1 = 0$

$$\left[-3\left(-\tfrac{1}{2}\right) + 5\left(-\tfrac{1}{2}\right)\right]c_1 = 0$$

$$-c_1 = 0, \quad c_1 = 0$$

LOOK AT OUR RECURRENCE:

$$2\left(k - \tfrac{3}{2}\right)\left(k - \tfrac{1}{2}\right)c_{k+1} + 5\left(k - \tfrac{1}{2}\right)c_{k+1} + c_{k-1} = 0, \quad k \geq 1$$

$$\left[(2k-3) + 5\right]\left(k - \tfrac{1}{2}\right)c_{k+1} = -c_{k-1}$$

$$c_{k+1} = \frac{-c_{k-1}}{(2k+2)\left(k - \frac{1}{2}\right)} = \frac{-c_{k-1}}{(k+1)(2k-1)}$$

SO:

$$c_2 = \frac{-c_0}{2}$$

$$c_3 = \frac{-c_1}{\text{STUFF}} = 0 \quad (c_1 = 0)$$

AGAIN ALL ODD c's $= 0$

$$c_4 = \frac{-c_2}{4\cdot 5} = \frac{c_0}{2\cdot 4\cdot 5}$$

$$c_6 = \frac{-c_4}{6\cdot 9} = \frac{-c_0}{2\cdot 4\cdot 5\cdot 6\cdot 9}$$

$$y_2 = c_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4\cdot 5}x^4 - \frac{1}{2\cdot 4\cdot 5\cdot 6\cdot 9}x^6 + \dots\right]$$

WE MIGHT DO ANOTHER EXAMPLE IN CLASS BUT I DON'T FEEL LIKE WRITING ONE UP HERE. INSTEAD, A RECAP/SUMMARY OF THE METHOD.

1. PUT DIFFY Q IN STANDARD FORM $y'' + P(x)y' + Q(x)y = 0$
2. IF $x=0$ IS NONSINGULAR USE $y = \sum_{n\geq 0} c_n x^n$ AS IN LAST LECTURE (GETS US 2 L.I. SOLUTIONS)
3. IF $x=0$ IS A REGULAR SINGULAR POINT, USE
$$\left.\begin{aligned} y &= \sum_{n\geq 0} c_n x^{n+r} \\ y' &= \sum_{n\geq 0} (n+r) c_n x^{n+r-1} \\ y'' &= \sum_{n\geq 0} (n+r-1)(n+r) c_n x^{n+r-2} \end{aligned}\right\} \text{NOTE: ALL 3 START @ } n=0$$
4. MULTIPLY IN ANY POLYNOMIAL COEFFICIENTS INTO YOUR SERIES ONCE YOU PLUG IN.
5. FACTOR OUT x^r (CANCEL IF YOU WANT)
6. COMBINE POWER SERIES
7. LOOK AT THE LOWEST POWER OF x COEFFICIENT $= 0$. THIS WILL GIVE YOU THE INDICIAL ROOTS r_1 & r_2 (CHECK USING THE QUICK WAY)
8. USE r_1 & r_2 TO DETERMINE THE 2 RECURRENCE RELATIONS AND FIND 2 L.I. SOLUTIONS.

OK THAT WAS AWFUL SO LET'S MOVE ON TO A GREAT TOPIC

VECTOR CALCULUS

WE'RE GOING TO BE DEALING WITH A LOT OF PARAMETRIZED CURVES IN WHAT FOLLOWS. THESE ARE JUST "PATHS" IN $\mathbb{R}^n$, MORE SPECIFICALLY FUNCTIONS $r(t) : \mathbb{R} \longrightarrow \mathbb{R}^n$

TAKES A REAL NUMBER AND GIVES YOU A VECTOR

IN $\mathbb{R}^2$ WE'LL WRITE

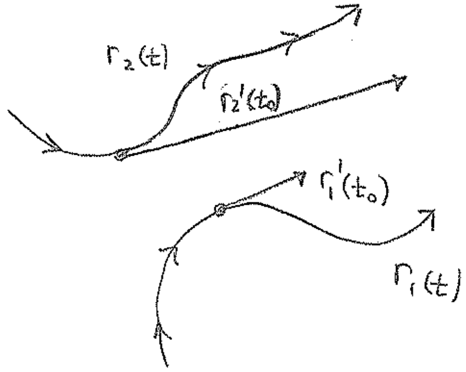
$$r(t) = (x(t), y(t))$$

IN $\mathbb{R}^3$: $r(t) = (x(t), y(t), z(t))$

WHEN WE TAKE THE DERIVATIVE, WE JUST TAKE IT COMPONENTWISE:

$$r'(t) = (x'(t), y'(t), z'(t))$$ (PRIME IS DERIVATIVE WITH RESPECT TO t)

IF WE CONSIDER $r(t)$ AS DESCRIBING THE MOTION OF A PARTICLE AT TIME t, $r'(t)$ IS THE VELOCITY VECTOR OF THAT MOTION. IN PARTICULAR $r'(t)$ IS TANGENT TO THE CURVE.



THE GREATER THE MAGNITUDE OF THE VECTOR, THE FASTER THE PARTICLE IS MOVING.

EX: THE PARTICLE WHOSE PATH IS $r_2(t)$ TO THE LEFT IS GOING FASTER THAN THE PARTICLE WHOSE PATH IS $r_1(t)$.

$r'(t)$ IS SOMETIMES DENOTED V, AND THE SPEED IS:

$$\|r'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

TO FIND THE LENGTH OF A PATH $r(t)$, CALLED THE ARCLENGTH (COMMONLY DENOTED "S"), WE USE THE FORMULA:

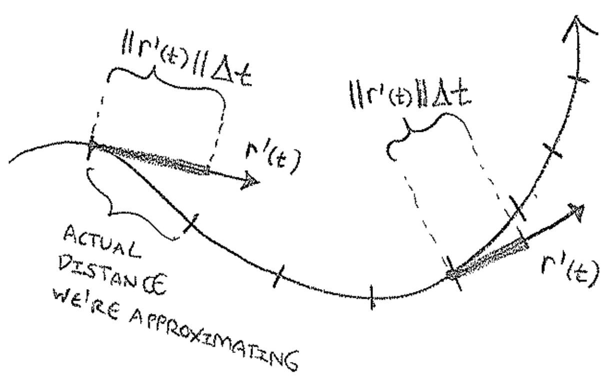
$$S = \int_{t_0}^{t_1} \|r'(t)\|\, dt$$

COMPUTES THE ARCLENGTH FROM $t = t_0$ TO $t = t_1$ OF $r(t)$.

THIS MAKES SENSE SINCE WE CAN APPROXIMATE THE ARCLENGTH BY CHOPPING UP OUR CURVE AND ESTIMATING EACH SMALL PIECES LENGTH BY:

$$\Delta S = \|r'(t)\|\, \Delta t \qquad (\text{DISTANCE} = \text{RATE} \cdot \text{TIME})$$

THEN WE TAKE A LIMIT AS THE # OF "PIECES" WE USE GOES TO ∞ TO GET OUR INTEGRAL.

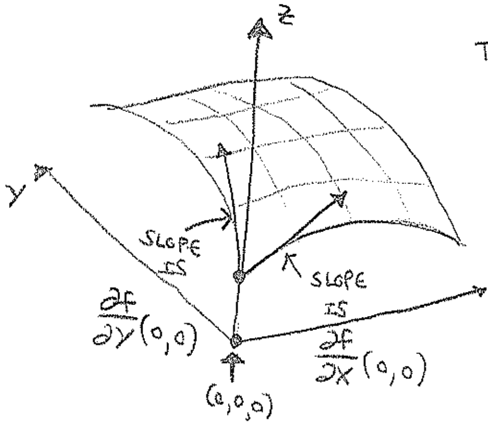


PARTIAL DERIVATIVES

IF $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ (IN OTHER WORDS A REAL VALUED FUNCTION OF n VARIABLES) WE CAN TAKE DERIVATIVES WITH RESPECT TO ANY OF THE n VARIABLES WHILE KEEPING THE OTHERS CONSTANT (THIS IS THE PARTIAL DERIVATIVE)

GEOMETRICALLY

SUPPOSE $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$. WE CAN THINK OF f AS ASSOCIATING A Z-COORDINATE TO ANY (x,y) PAIR, THUS PICTURING THE SET OF POINTS $(x, y, f(x,y)) \in \mathbb{R}^3$ AS A SURFACE.

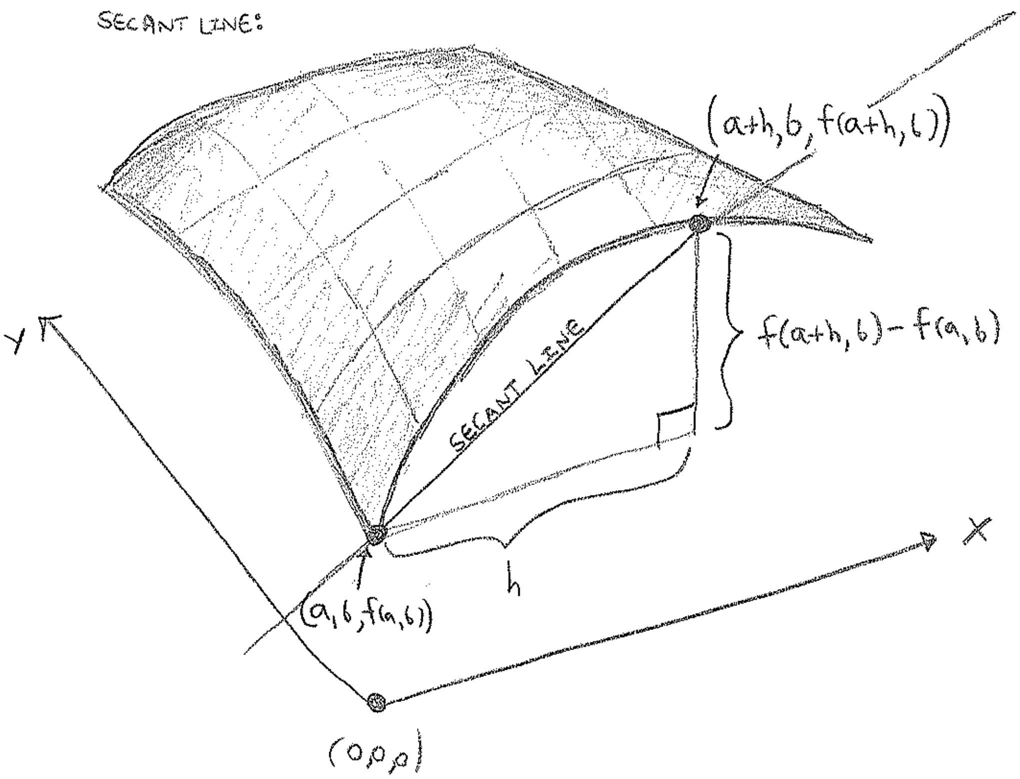


THEN $\frac{\partial f}{\partial x}(0,0)$ IS THE SLOPE OF THE TANGENT LINE TO THIS SURFACE AT THE POINT $(0,0,f(0,0))$ IN THE X-DIRECTION. THINK OF IT AS SLICING OUR SURFACE ALONG $Y=0$ AND LOOKING AT THE SLOPE ALONG THE RESULTING CURVE.

FORMALLY

THE DEFINITIONS OF $\frac{\partial f}{\partial x}(a,b)$ $\left(\text{AND } \frac{\partial f}{\partial y}(a,b)\right)$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \underbrace{\frac{f(a+h,b) - f(a,b)}{h}}_{\text{SLOPE OF SECANT LINE:}}, \qquad \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



EX: $f = xe^{z}\cos y + z^2 \sin(xy)$

FIND $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = -xe^{z}\sin y + z^2\cos(xy)\,x \qquad (x \,\&\, z \text{ ARE CONSIDERED CONSTANTS})$$

<u>NOTATION</u> INSTEAD OF $\frac{\partial f}{\partial y}$ WE USUALLY WRITE f_y.

INSTEAD OF $\frac{\partial^2 f}{\partial y^2}$ (2ND PARTIAL W/ RESPECT TO y) WE WRITE f_{yy}

INSTEAD OF $\frac{\partial^2 f}{\partial x \partial y}$ WRITE f_{yx}

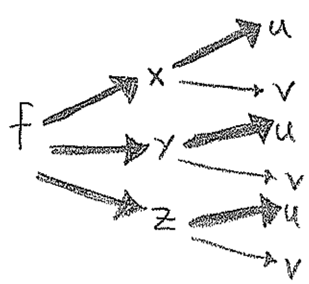
<u>RMK</u> IF A FUNCTION HAS CONTINUOUS 2ND PARTIAL DERIVATIVES, THE "MIXED PARTIALS" ARE ALWAYS EQUAL. IN OTHER WORDS, $f_{xy} = f_{yx}$, $f_{xzy} = f_{yxz}$ ETC.

NOW SUPPOSE:

EX: $f = xy + e^{z}\cos x$, $x = uv$, $y = u^2$, $z = 3 - v$

FIND $\frac{\partial f}{\partial u}$.

WE COULD JUST SUBSTITUTE IN FOR x, y, z IN TERMS OF u, v. INSTEAD CONSIDER THE FOLLOWING DIAGRAM SHOWING WHAT IS A FUNCTION OF WHAT:



A CHANGE IN u AFFECTS f THREE DIFFERENT WAYS (3 DIFFERENT PATHS FROM f TO u). FOR EACH PATH WE ADD A CORRESPONDING TERM TO OUR PARTIAL DERIVATIVE:

$$\frac{\partial f}{\partial u} = \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}}_{\text{TOP PATH}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \underbrace{\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}}_{\text{BOTTOM PATH}}$$

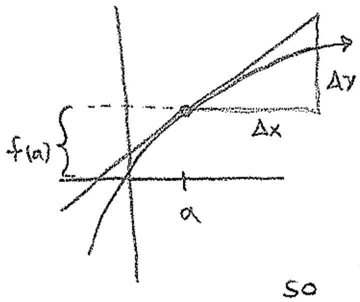
SO $\frac{\partial f}{\partial u} = (y - e^{z}\sin x)(v) + (x)(2u) + (e^{z}\cos x)(-1)$ PUT IN TERMS u & v:

$$= \left[u^2 - e^{(3-v)}\sin(uv)\right]v + 2u^2 v - e^{(3-v)}\cos(uv)$$

THIS MAKES TAKING DERIVATIVES MUCH EASIER!

TANGENT LINES/PLANES/HYPERPLANES

IN CALC I WE APPROXIMATED FUNCTIONS OF 1 VARIABLE W/ A TANGENT LINE:



$$\frac{\Delta y}{\Delta x} = f'(a)$$

SO $\Delta y = f'(a)\Delta x$ IS OUR ESTIMATED CHANGE IN THE FUNCTION IF Δx IS OUR X DISPLACEMENT, $(x-a)$

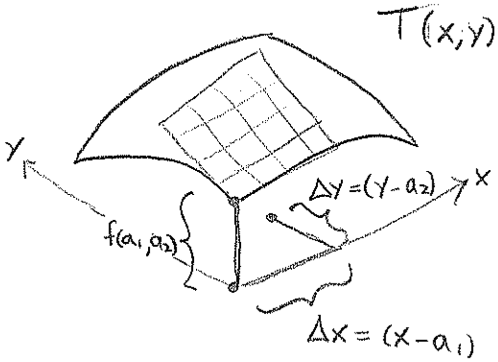
SO $T(x) = f(a) + f'(a)(x-a)$ IS OUR TANGENT LINE

FOR FUNCTIONS $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ WE CONSTRUCTED A <u>TANGENT PLANE</u> SIMILARLY

TANGENT PLANE AT (a_1, a_2) IS:

$$T(x,y) = f(a_1,a_2) + \underbrace{\frac{\partial f}{\partial x}(a_1,a_2)\left(x-a_1\right)} + \underbrace{\frac{\partial f}{\partial y}(a_1,a_2)\left(y-a_2\right)}$$

ESTIMATED Δf AS WE MOVE $x-a_1$ IN X-DIRECTION | " " $y-a_2$ IN Y-DIRECTION



FOR FUNCTIONS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ AS YOU MAY HAVE GUESSED WE GET:

TANGENT "PLANE" (USUALLY CALLED A HYPERPLANE) AT $(a_1, a_2, \dots, a_n)$:

$$T(x_1, x_2, \dots, x_n) = f(a_1, \dots, a_n) + \frac{\partial f}{\partial x_1}(x_1 - a_1) + \frac{\partial f}{\partial x_2}(x_2 - a_2) + \dots + \frac{\partial f}{\partial x_n}(x_n - a_n)$$

$$= f(a_1, \dots, a_n) + \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix}$$

(ALL DERIVS ARE EVALUATED AT $(a_1, a_2, \dots, a_n)$)

MATRIX MULTIPLICATION

NOW FOR FUNCTIONS THAT ARE VECTOR VALUED, $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

SO f LOOKS LIKE $\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$ WE CAN DO A SIMILAR APPROXIMATION USING PARTIALS:

$$T(a_1,\dots,a_n) = \underset{m\times 1 \text{ VECTOR}}{f(a_1,\dots,a_n)} + \underbrace{\underset{m\times n}{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}} \underset{n\times 1}{\begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix}}}_{\text{ESTIMATED CHANGE IN } f \text{ (A VECTOR)}}$$

(AGAIN ALL $\frac{\partial f}{\partial x}$'S ARE EVALUATED AT $(a_1,\dots,a_n)$)

LINEAR ALGEBRA IS EVERYWHERE!

AGAIN, THIS MATRIX WHEN MULTIPLIED BY A VECTOR CORRESPONDING TO SOME CHANGES IN THE VARIABLES $x_1 \dots x_n$ GIVES YOU A VECTOR ESTIMATING BY HOW MUCH f CHANGES IN THAT DIRECTION. THIS MAKES SENSE IF YOU MULTIPLY IT OUT. THE FIRST ELEMENT IS:

$$\frac{\partial f_1}{\partial x_1}(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(x_2 - a_2) + \dots + \frac{\partial f_1}{\partial x_n}(x_n - a_n)$$

WHICH IS JUST AN ESTIMATED CHANGE IN f_1.

DIRECTIONAL DERIVATIVES

IF $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ IS A REAL VALUED FUNCTION, AND $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ IS SOME UNIT VECTOR, WE CAN ESTIMATE HOW MUCH f CHANGES IN THE DIRECTION OF u AT A POINT (a_1, a_2, a_3) BY TAKING:

$$\Delta f \approx \begin{pmatrix} \frac{\partial f}{\partial x}(a_1,a_2) & \frac{\partial f}{\partial y}(a_1,a_2) & \frac{\partial f}{\partial z}(a_1,a_2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{BY ABOVE}$$

$$= \frac{\partial f}{\partial x}(a_1,a_2)\, u_1 + \frac{\partial f}{\partial y}(a_1,a_2)\, u_2 + \frac{\partial f}{\partial z}(a_1,a_2)\, u_3$$

THE USUAL NOTATION IS $D_u f(a_1, a_2, a_3)$ FOR THE DIRECTIONAL DERIVATIVE IN u DIRECTION. ALSO NOTE THIS SAME DEFINITION MAKES SENSE FOR ANY # OF VARIABLES.