LAST TIME: WE TALKED ABOUT PARTIAL DERIVATIVES AND SAID WE COULD USE A MATRIX OF THESE PARTIAL DERIVATIVES TO ESTIMATE CHANGES IN FUNCTIONS WHEN WE CHANGE THE VARIABLES. MORE SPECIFICALLY:

IF $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ IS A FUNCTION W/ ALL PARTIAL DERIVATIVES EXISTING THEN WE CAN USE THE MATRIX:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \longleftarrow m \times n$$

THIS IS CALLED THE <u>TOTAL DERIVATIVE</u> AND DENOTED Df.

TO ESTIMATE CHANGES IN $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$ (VECTOR VALUED) BY MULTIPLYING BY A VECTOR

WHICH IS OF THE FORM $\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{pmatrix} \longleftarrow n \times 1$ (SO WE CAN MULTIPLY BY Df ON THE LEFT)

IN OUR CASE WE'LL ONLY DEAL WITH FUNCTIONS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ SO

$$Df = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \cdots \quad \frac{\partial f}{\partial x_n} \right)$$

THIS VECTOR IS COMMONLY CALLED THE <u>GRADIENT</u> AND DENOTED ∇f.

WE ALSO USUALLY WRITE IT WITH COMMAS IN BETWEEN THE PARTIALS.

EX: $f = x\cos(e^z) + y\ln(x-z)$ <u>FIND ∇f</u>

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \left(\cos(e^z) + \frac{y}{x-z},\ \ln(x-z),\ -x\sin(e^z)e^z - \frac{y}{x-z} \right)$$

EX: $f = xyz + e^y x^2$

ESTIMATE HOW MUCH f CHANGES AT THE POINT $(1,0,-1)$ IF WE CHANGE x BY 1, y BY -2, z BY 0. (i.e. USE THE "TANGENT PLANE" WE TALKED ABOUT LAST TIME)

BY WHAT WE SAID ABOVE, WE HAVE:

$$\Delta f \approx \left(\frac{\partial f}{\partial x}(1,0,-1) \quad \frac{\partial f}{\partial y}(1,0,-1) \quad \frac{\partial f}{\partial z}(1,0,-1) \right) \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$= \left(yz + 2xe^y \Big|_{(1,0,-1)} \quad xz + e^y x^2 \Big|_{(1,0,-1)} \quad xy \Big|_{(1,0,-1)} \right) \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$= (2 \quad 0 \quad 0) \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$= 2(1) + 0(-2) + 0(0) = 2$$ IS OUR ESTIMATED Δf

NOW WE DEFINED A DIRECTIONAL DERIVATIVE AS THIS SAME ESTIMATED CHANGE IN f

BUT OUR $\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$ VECTOR HAD TO BE A UNIT VECTOR (LENGTH 1).

EX: $f = xyz + e^y x^2$. FIND THE DIRECTIONAL DERIVATIVE AT THE POINT $(1,0,-1)$ IN THE DIRECTION OF THE VECTOR $V = (1,-2,0)$.

(THIS IS THE SAME f AND DIRECTION AS LAST TIME)

SINCE V IS NOT A UNIT, WE SCALE IT SO THAT IT IS:

$\|V\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$, $u = \frac{V}{\|V\|} = \frac{1}{\sqrt{5}}(1,-2,0)$ IS A UNIT

$$\Delta f \approx (2 \quad 0 \quad 0) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \frac{2}{\sqrt{5}}$$

(WE DID THIS ABOVE)

NOTICE THAT OUR DIRECTIONAL DERIVATIVE IS JUST

$$\left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \nabla f \cdot u$$

DOT PRODUCT

WE WILL USE THIS NOTATION FROM NOW ON.

IN PARTICULAR, IF $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ THEN $D_u f = \nabla f \cdot u = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot (1,0,0) = \frac{\partial f}{\partial x}$

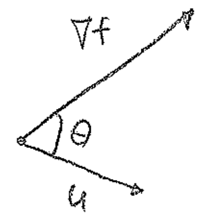
$u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ THEN $D_u f = \frac{\partial f}{\partial y}$

$u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ THEN $D_u f = \frac{\partial f}{\partial z}$

SO PARTIAL DERIVATIVES ARE JUST THE DIRECTIONAL DERIVATIVES IN THE X, Y, & Z DIRECTIONS. ALSO RECALL THAT:

$$D_u f = \nabla f \cdot u = \| \nabla f \| \underbrace{\| u \|}_{=1} \cos\theta = \| \nabla f \| \cos\theta$$

θ = ANGLE BETWEEN ∇f & u

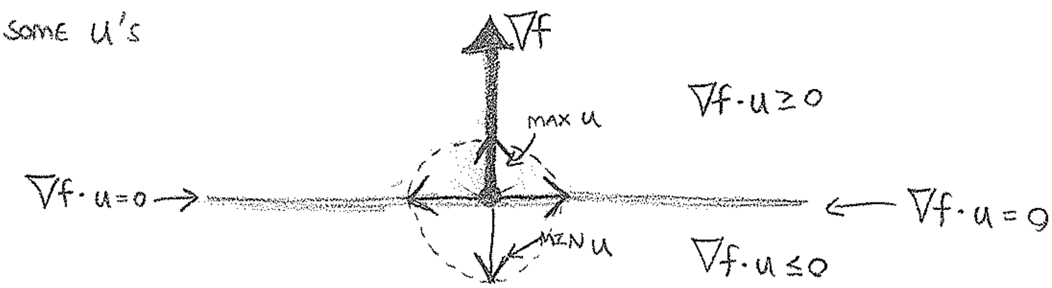


SINCE $\cos\theta$ IS AT MOST 1 AND AT LEAST -1, THIS SHOWS THAT THE DIRECTIONAL DERIVATIVE IS GREATEST WHEN $\cos\theta = 1$ $(\theta = 0)$ AND THE DIRECTIONAL DERIVATIVE IS SMALLEST WHEN $\cos\theta = -1$ $(\theta = \pi)$.

SO:

1. $D_u f$ IS OF MAXIMUM VALUE $\| \nabla f \|$ WHEN u IS IN THE DIRECTION OF ∇f
2. $D_u f$ IS OF MINIMUM VALUE $-\| \nabla f \|$ WHEN u IS IN THE DIRECTION OF $-\nabla f$

GRAPH OF SOME u'S AND ∇f:



FOR THIS REASON WE SAY THAT ∇f IS THE "DIRECTION OF GREATEST INCREASE" OF f.
$-\nabla f$ IS THE "DIRECTION OF GREATEST DECREASE" OF f.

DEF: IF $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ IS SOME FUNCTION, A LEVEL CURVE (OR LEVEL SURFACE...) OF f IS A SET OF POINTS IN $\mathbb{R}^n$ ON WHICH f TAKES THE SAME VALUE. (I.E. f IS CONSTANT)

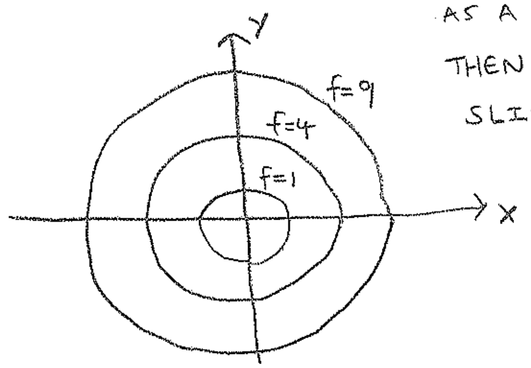
EX: DESCRIBE THE LEVEL CURVES OF $f(x,y) = x^2 + y^2$

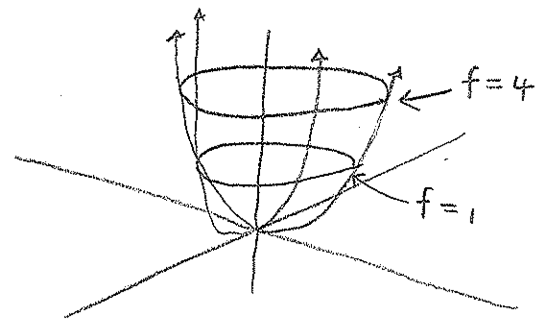
THE LEVEL CURVE CORRESPONDING TO $f = 1$ IS $1 = x^2 + y^2$ UNIT CIRCLE

$f = 4$ IS $4 = x^2 + y^2$ RADIUS 2 CIRCLE

$f = 9$ IS $9 = x^2 + y^2$ RADIUS 3 CIRCLE

SO THE LEVEL CURVES ARE CIRCLES. IF WE THINK OF THE GRAPH OF THIS FUNCTION AS A SURFACE (i.e. THE POINTS $(x, y, f(x,y))$ IN $\mathbb{R}^3$) THEN THESE LEVEL SURFACES ARE JUST THE "HORIZONTAL SLICES" IN OUR SURFACE:





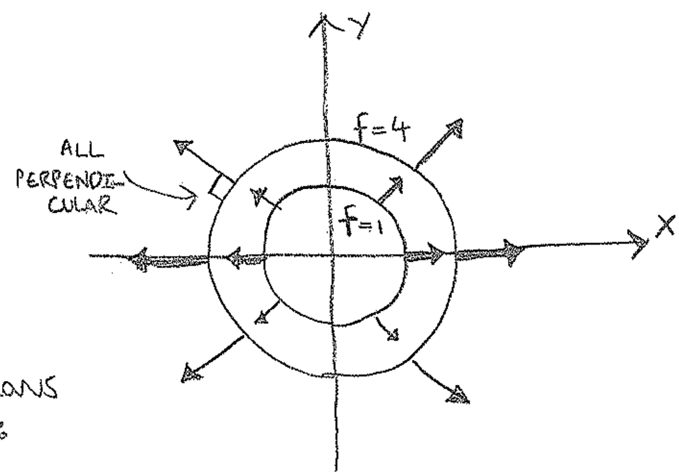
FACT: $\boxed{\nabla f \text{ IS ALWAYS } \perp \text{ TO THE LEVEL SURFACES OF } f.}$

LOOK IN OUR ABOVE EXAMPLE. HERE $\nabla f = (2x, 2y)$. WE CAN PICTURE ∇f AS A VECTOR AT EVERY POINT IN $\mathbb{R}^2$, CALLED A VECTOR FIELD.

ALONG THE X-AXIS $(y=0)$ $\nabla f = (2x, 0)$

Y-AXIS $(x=0)$ $\nabla f = (0, 2y)$

PLOTTING THESE ON OUR LEVEL CURVES PICTURE:



NOW SOME APPLICATIONS OF GRADIENTS:

EX: SUPPOSE THE TEMPERATURE IN $\mathbb{R}^3$ IS GIVEN BY THE FUNCTION

$$T(x,y,z) = xe^z + y^2 + xyz$$

SUPPOSE BOB THE BUG IS SITTING AT $(1,1,1)$ AND WANTS TO COOL DOWN THE QUICKEST SINCE PHILLY IS HOT AND HUMID. IN WHAT DIRECTION SHOULD HE GO?

$\nabla T = (e^z + yz,\ 2y + xz,\ xe^z + xy)$ IS OUR GRADIENT.

$\nabla T(1,1,1) = (e+1,\ 3,\ e+1)$ IS THE GRADIENT WHERE BOB IS.

TO SEE HOW MUCH T CHANGES IN ANY DIRECTION, WE USE DIRECTIONAL DERIVATIVES.

SINCE $-\nabla T$ IS THE DIRECTION OF GREATEST <u>DECREASE</u>, HE SHOULD TRAVEL IN THAT DIRECTION. SO, IN THE DIRECTION OF:

$$-\nabla T(1,1,1) = (-e-1, -3, -e-1)$$

EX: FIND THE TANGENT PLANE TO THE SURFACE $z = xy + y^2 - x^2$ AT THE POINT $(1,2,5)$.

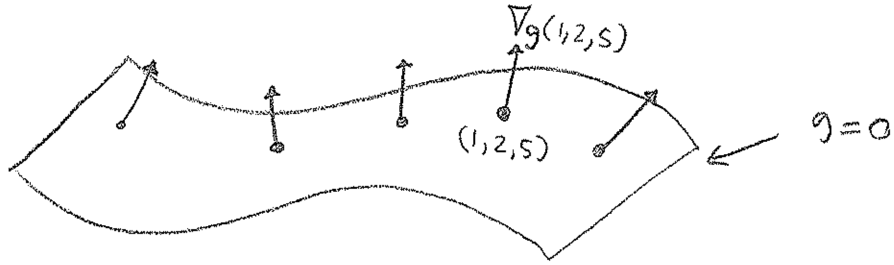
HERE WE USE OUR OTHER PROPERTY OF GRADIENTS.

LET $g(x,y,z) = z - xy - y^2 + x^2$

SO OUR SURFACE IS A LEVEL SURFACE OF g CORRESPONDING TO $g=0$.

THUS ∇g WE BE A VECTOR FIELD ON <u>ALL OF $\mathbb{R}^3$</u>, BUT ON OUR SURFACE $(g=0)$ IT WILL BE PERPENDICULAR.

$$\nabla g = (-y + 2x, -x - 2y, 1)$$



THUS

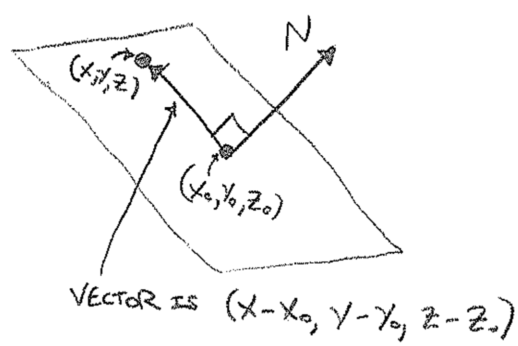
$\nabla g(1,2,5) = (0,-5,1)$ WILL BE A NORMAL VECTOR TO OUR SURFACE & THUS A NORMAL VECTOR FOR OUR TANGENT PLANE. TO DEFINE A PLANE YOU ONLY NEED:

1. A POINT (x_0, y_0, z_0) ON IT
2. A NORMAL VECTOR $N = (N_1, N_2, N_3)$

THEN THE PLANE IS THE SET OF ALL (x,y,z) s.t. THE VECTOR $(x-x_0, y-y_0, z-z_0)$ IS $\perp$ TO N:

$$N \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$N_1(x-x_0) + N_2(y-y_0) + N_3(z-z_0) = 0$$



SO IN OUR CASE OUR $N = (0,-5,1)$, $(x_0, y_0, z_0) = (1,2,5)$:

$$0(x-1) - 5(y-2) + (z-5) = 0$$

$$z - 5y = -5$$

$f = xy + y^2 - x^2$

NOTE THAT SINCE WE HAD A SURFACE OF THE FORM $z = f(x,y)$ WE COULD JUST USE THE FORMULA:

$$z = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2) \qquad \frac{\partial f}{\partial x} = y - 2x = 0$$

$$z = 5 + 0(x-1) + 5(y-2) \qquad \frac{\partial f}{\partial y} = x + 2y = 5$$

$$z = 5y - 5$$

$$z - 5y = -5$$

BUT IF IT IS NOT OF THE FORM $z =$ STUFF ONLY IN x & y WE NEED TO USE ∇'s.

EX: FIND THE TANGENT PLANE TO THE SURFACE $e^{x+y} - x^2 e^z + z^3 x = 8$ AT THE POINT $(1,1,2)$

LET $g(x,y,z) = e^{x+y} - x^2 e^z + z^3 x - 8$ SO OUR SURFACE IS THE LEVEL SURFACE $g = 0$

$$\nabla g = \left(e^{x+y} - 2xe^z + z^3,\ e^{x+y},\ -x^2 e^z + 3z^2 x\right)$$

$$\nabla g(1,1,2) = \left(e^2 - 2e^2 + 8,\ e^2,\ -e^2 + 12\right)$$

$$= \left(-e^2 + 8,\ e^2,\ -e^2 + 12\right) \quad (= N)$$

AND OUR TANGENT PLANE IS THUS:

$$(-e^2 + 8)(x-1) + (e^2)(y-1) + (-e^2 + 12)(z-2) = 0$$

DIVERGENCE & CURL

EARLIER WE DEFINED THE GRADIENT ∇f OF A FUNCTION AND VIEWED IT AS A VECTOR AT EVERY POINT IN $\mathbb{R}^n$ OR VECTOR FIELD. VECTOR FIELDS ARISE IN ALL SORTS OF REAL LIFE APPLICATIONS (GRAVITATIONAL FIELDS, ELECTRIC FIELDS, FIELDS DESCRIBING FLUID FLOW). WE INTRODUCE SOME USEFUL OPERATIONS ON VECTOR FIELDS THAT WILL GIVE US SOME GEOMETRIC INTUITION AS TO HOW OUR FIELDS LOOK OR BEHAVE.

VECTOR FIELDS ARE USUALLY DENOTED F AND WRITTEN IN A FEW DIFFERENT WAYS:

$F = (P, Q, R)$ ($P, Q, \& R$ A FUNCTIONS OF x, y AND z)

$F = P\,i + Q\,j + R\,k$

CURL

IF $F = (P,Q,R)$ IS SOME VECTOR FIELD IN $\mathbb{R}^3$, WE DEFINE THE CURL OF F DENOTED $\operatorname{curl} F$

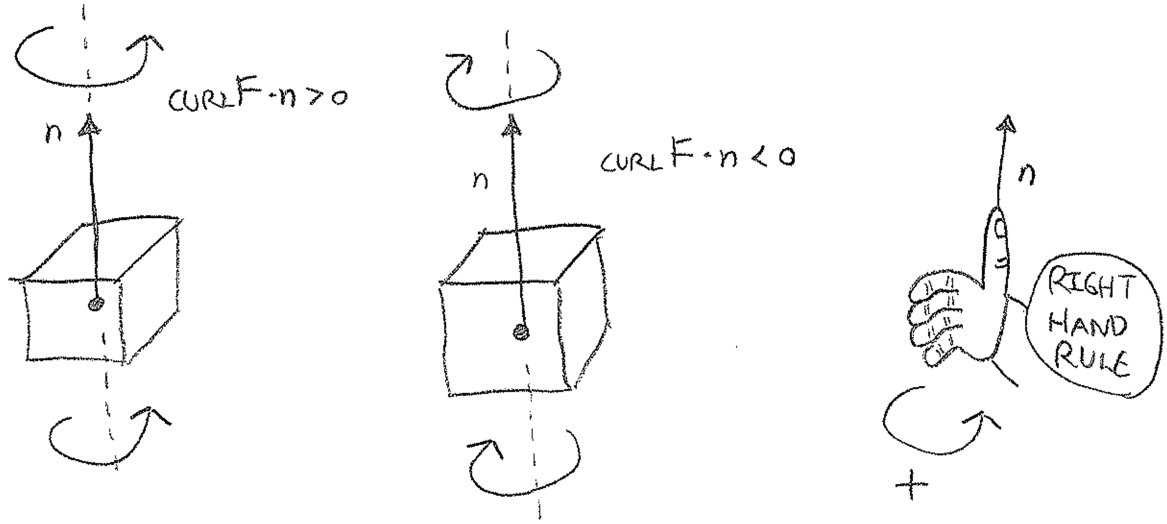
$$\text{BY:}\quad \operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$ BY WHICH WE MEAN THE VECTOR

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

NOTE THAT $\operatorname{curl} F$ IS A VECTOR.

THE VECTOR $\operatorname{curl} F$ TELLS US HOW MUCH OUR VECTOR FIELD F ROTATES ABOUT ANY AXIS AT A POINT. MORE PRECISELY:

FOR ANY UNIT VECTOR n, $\operatorname{curl} F \cdot n$ MEASURES HOW MUCH F ROTATES ABOUT THE AXIS ALONG n.



SO IMAGINE AN INFINITELY SMALL BOX (I.E. A POINT). THEN $\operatorname{curl} F \cdot n$ MEASURES HOW FAST THIS BOX WILL SPIN AROUND THE AXIS CONTAINING n.

$\operatorname{curl} F \cdot n > 0$ MEANS IT'S SPINNING IN THE DIRECTION THAT YOUR FINGERS CURL ON YOUR RIGHT HAND WHEN YOUR THUMB POINTS ALONG n.

SO IN PARTICULAR, THE X, Y, & Z COMPONENTS OF CURL F WHICH ARE:

$\text{curl}\, F \cdot i$, $\text{curl}\, F \cdot j$, $\text{curl}\, F \cdot k$

MEASURE HOW MUCH THE FIELD ROTATES AROUND THE X, Y, AND Z COORDINATE DIRECTIONS.

EX: $F = (xy, zx, yz)$ FIND CURL F

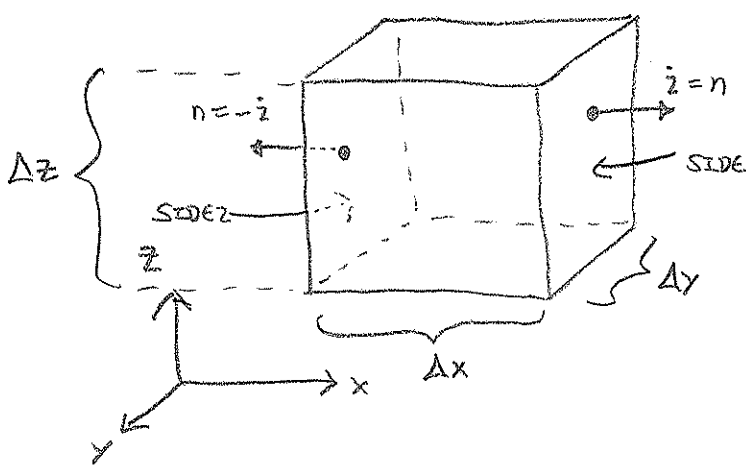
$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & zx & yz \end{vmatrix} = (z-x)i - (0-0)j + (z-x)k$$
$$= (z-x, 0, z-x)$$

SINCE $\text{curl}\, F \cdot j = 0$ <u>EVERYWHERE</u>, OUR INFINITELY SMALL BOXES ALL DO NOT ROTATE AROUND THE Y-DIRECTION.

DIVERGENCE

THE DIVERGENCE OF A VECTOR FIELD F MEASURES HOW MUCH "FLOWS" INSIDE OR OUTSIDE OF ONE OF OUR INFINITELY SMALL BOXES.

TO FIGURE OUT WHAT THIS MIGHT BE, LETS DRAW SOME BOX:



SUPPOSE $F = (P, Q, R)$

COMPONENT OF F IN THE n DIRECTION

WE DEFINE THE FLOW OF F ALONG ANY SIDE OF OUR BOX BY $(F \cdot n)(\text{AREA})$

SO THE FLOW (OR FLUX) ALONG THE TWO SIDES 1 & 2 LABELLED ABOVE IS:

$$(F \cdot i)(\Delta y \Delta z) + (F \cdot (-i))(\Delta y \Delta z) = P(x+\Delta x, y, z)\,\Delta y\, \Delta z - P(x,y,z)\,\Delta y\, \Delta z$$

NOTE: WE EVALUATE ARE FIELD AT DIFFERENT POINTS! (SHOULD MAKE PHYSICAL SENSE)

$$= \left[\frac{P(x+\Delta x, y, z) - P(x,y,z)}{\Delta x} \right] \Delta x \Delta y \Delta z \qquad (\text{MULT \& DIVIDE BY } \Delta x)$$

NOW AS OUR BOX $\longrightarrow 0$, $\Delta x \to dx$, $\Delta y \to dy$, $\Delta z \to dz$, AND THE BRACKETED PART IS $\frac{\partial P}{\partial x}$

$$\xrightarrow[\text{BOX} \to 0]{\text{LIM}} \frac{\partial P}{\partial x}\, dx\, dy\, dz$$

SO OUR FLUX IS (DOING THE SAME W/ THE OTHER TWO PAIRS OF SIDES)

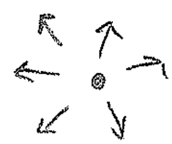
$$\text{FLUX} = \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\, dy\, dz$$

SO THE NUMBER IN ( )'S TELLS US HOW MUCH FLOW IN OR OUT OF OUR BOX WE HAVE.

DEF THE DIVERGENCE OF A FIELD $F = (P, Q, R)$ IS:

$$\text{DIV } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(SO DIVERGENCE IS A FLOW PER UNIT VOLUME OF AN INFINITELY SMALL BOX)

THE IDEA IS THAT DIV $F > 0$ LOOKS LIKE:  AROUND YOUR POINT

AND THAT DIV $F < 0$ LOOKS LIKE: 

EX: $F = (\underbrace{x^2 \sin(yz)}_{P},\ \underbrace{y \cos z}_{Q},\ \underbrace{z^2 \cos(xy)}_{R})$

FIND DIV F. DESCRIBE THE FLOW AROUND THE POINT $(0, 1, \pi)$

$$\text{DIV } F = 2x \sin(yz) + \cos z + 2z \cos(xy)$$

$$\text{DIV } F(0, 1, \pi) = -1 + 2\pi > 0$$

THUS AROUND OUR INFINITELY SMALL BOX AT $(0, 1, \pi)$, WE HAVE MORE FLOWING OUT THAN IN.