LAST TIME: WE TALKED ABOUT DIVERGENCE AND CURL OF A VECTOR FIELD F.

DIVERGENCE: "FLOW" IN OR OUT AT A POINT

CURL: HOW MUCH OUR VECTOR FIELD "SPINS" IN ANY DIRECTION AT A POINT

IF $F = (P, Q, R)$

$$\text{DIV}\,F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{(IS A NUMBER)}$$

$$\text{CURL}\,F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{(IS A VECTOR)}$$

LINE INTEGRALS

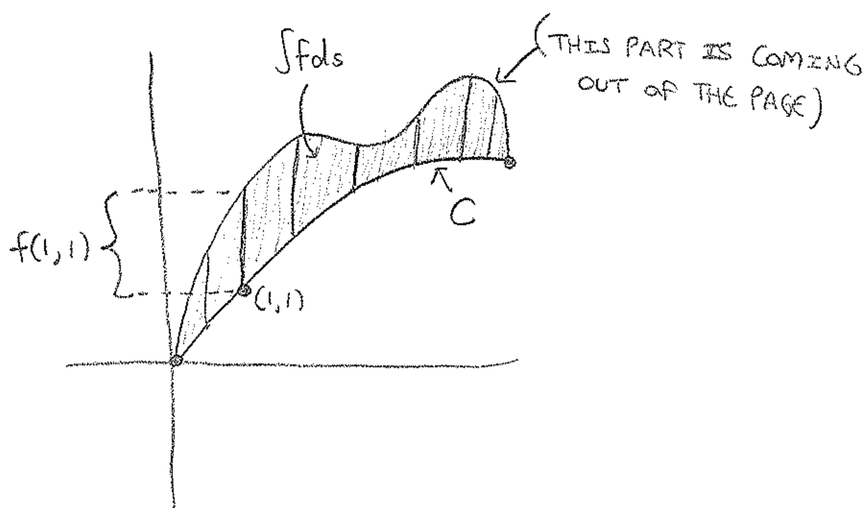
NOW WE MOVE ON TO A TYPE OF INTEGRAL WE CAN TAKE ALONG ANY PATH IN $\mathbb{R}^2$ OR $\mathbb{R}^3$.

LINE INTEGRALS IN $\mathbb{R}^2$ MAY LOOK LIKE:

$$\int f(x,y)\,dx, \quad \int f(x,y)\,dy, \quad \int f(x,y)\,\underset{\substack{\uparrow \\ \text{ARCLENGTH ELEMENT}}}{ds}$$

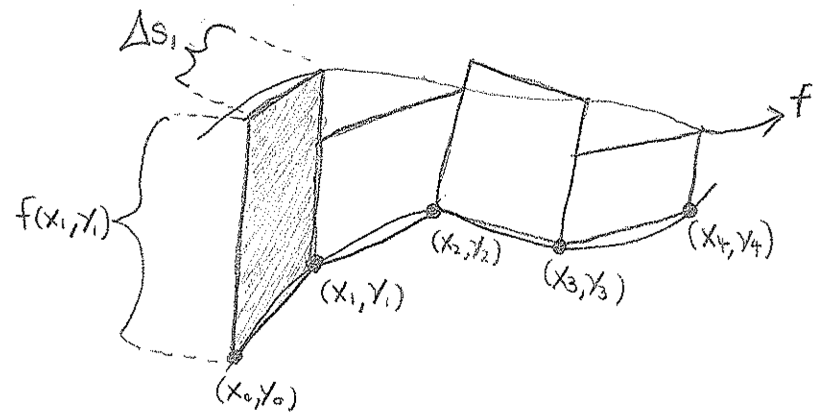
INTEGRATING AROUND SOME CURVE C

$\int f(x,y)\,ds$ IS THE AREA OF THE "CURTAIN" ABOVE OUR CURVE C WITH HEIGHT $f(x,y)$ AT EACH POINT:

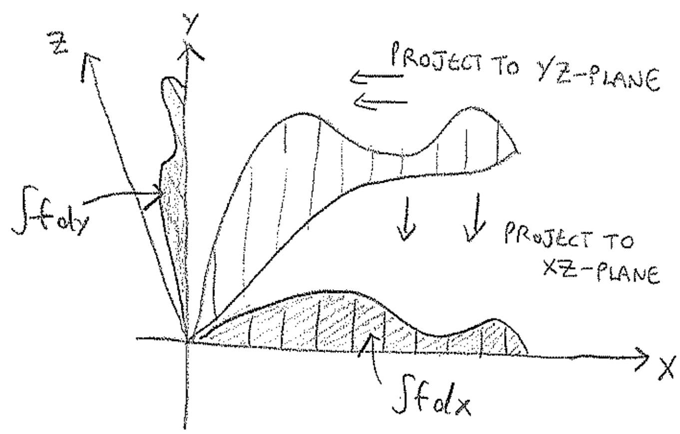


...SINCE IT IS COMPUTED BY TAKING OUR CURVE C & CHOPPING INTO PIECES, ESTIMATING Δs WITH A LINE SEGMENT AND ADDING UP:

$$\sum_i f(x_i, y_i)\,\Delta s_i$$



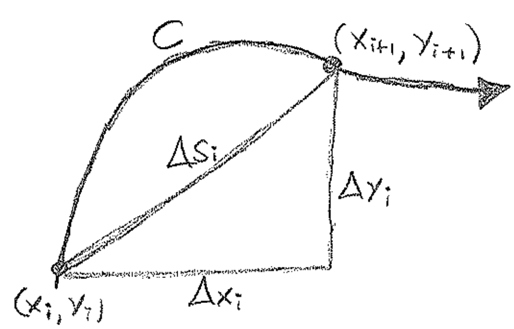
NOW THE INTEGRALS $\int f\,dx$, $\int f\,dy$ CAN BE THOUGHT OF AS AREAS AS WELL, BUT OF THE PROJECTIONS OF THESE "CURTAINS" WE FORM OVER OUR CURVES.



THIS IS BECAUSE WE COMPUTE $\int f\,dx$ WITH SUMS OF THE FORM

$$\sum f(x_i,y_i)\,\Delta x_i$$

AND Δx_i, Δy_i, Δs_i ARE RELATED BY THE Δ:



RMK: THE NOTATION $\int f\,dx + g\,dy$ IS COMMONLY USED & JUST MEANS $\int f\,dx + \int g\,dy$

NOW THAT WE CAN PICTURE THEM, LET'S LEARN HOW TO EVALUATE THEM.

1. PARAMETRIZE YOUR CURVE C : $r(t) = (x(t), y(t))$
2. PLUG IN! (THEN USE t LIMITS)

$$\int f(x,y)\,dx = \int f(x(t),y(t))\,\frac{dx}{dt}\,dt$$

$$\int f(x,y)\,dy = \int f(x(t),y(t))\,\frac{dy}{dt}\,dt$$

$$\int f(x,y)\,ds = \int f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

HERE WE USE THE FACT THAT $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$ FROM OUR Δ WE DREW

$$\left(\frac{\Delta s}{\Delta t}\right) = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}$$

LET INTERVALS GO TO ZERO...

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

EX: COMPUTE $\int xy\,dx + y^2\,dy$ ALONG THE UNIT CIRCLE ↺ IN THAT DIRECTION

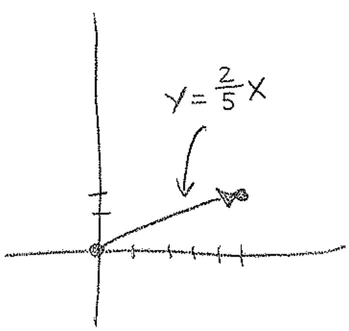
$r(t) = (\underset{x}{\cos t}, \underset{y}{\sin t})$ IS THE UNIT CIRCLE

$$\int xy\,dx + y^2\,dy = \int_0^{2\pi} \cos t \sin t\,\overbrace{(-\sin t\,dt)}^{dx} + \sin^2 t\,\overbrace{(\cos t\,dt)}^{dy}$$

$$= \int_0^{2\pi} -\cos t \sin^2 t + \cos t \sin^2 t\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$

EX: COMPUTE $\int \cos x\,dy$ ON THE STRAIGHT LINE FROM $(0,0)$ TO $(5,2)$



LET $t = x$ SO:

$$r(t) = \left(t, \frac{2}{5}t\right)$$

IS OUR PATH FROM $t=0$ TO 5

$x = t \qquad y = \frac{2}{5}t$

$dx = dt \qquad dy = \frac{2}{5}dt$

$$\int \cos x \, dy = \int_0^5 \cos t \left(\tfrac{2}{5} dt\right) \quad \swarrow dy$$

$$= \frac{2}{5}\Big[\sin t\Big]_0^5$$

$$= \frac{2}{5}\sin 5$$

EX: $\int (3+xy)\,ds$ ALONG THE CIRCLE RADIUS 3 FROM $(3,0)$ TO $(0,3)$

$r(t) = (\underbrace{3\cos t}_{x},\ \underbrace{3\sin t}_{y}) \qquad t = 0$ TO $\frac{\pi}{4}$

$$ds = \sqrt{(x')^2+(y')^2}\,dt$$

$$= \sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt$$

$$= 3\sqrt{\sin^2 t+\cos^2 t}\,dt$$

$$= 3dt$$

$$\int (3+xy)\,ds = \int_0^{\frac{\pi}{2}} (3+9\cos t\sin t)\,3\,dt$$

$$= 9\int 1+3\cos t\sin t\,dt$$

$$= 9\left(\tfrac{\pi}{2}-0\right) + 27\int \cos t\sin t\,dt$$

$u = \sin t$

$du = \cos t\,dt$

$$= \frac{9\pi}{2} + 27\int u\,du$$

$$= \frac{9\pi}{2} + 27\left[\tfrac{1}{2}(\sin^2 t)\right]\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{2} + \frac{27}{2}[1-0]$$

$$= \frac{9\pi}{2} + \frac{27}{2}$$

WORK INTEGRALS

IN PHYSICS WE DEFINE THE WORK DONE BY A FORCE AS

$$W = (F \cdot u)\,\Delta d$$

WHERE u IS A UNIT VECTOR IN THE DIRECTION OF MOTION

Δd IS THE DISPLACEMENT IN THAT DIRECTION

F IS OUR APPLIED FORCE VECTOR

$\cdot$ IS DOT PRODUCT

$(F \cdot u)$: FORCE IN THE DIRECTION u

IN OUR CASE WE THINK OF $r(t)$ AS A PATH OF SOME PARTICLE AT TIME t, AND F AS A FORCE (SAY GRAVITATION) ACTING ON IT. THEN:

$$W = \int F \cdot dr \qquad \text{WHERE } dr = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt \quad \left(\frac{dr}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)\right)$$

$$= \int F \cdot \frac{dr}{dt}\, dt \qquad \text{FOR COMPACTNESS WE WRITE } r'(t) = \frac{dr}{dt}$$

$$= \int F \cdot \underbrace{\frac{r'(t)}{\|r'(t)\|}}_{\substack{\text{UNIT VECTOR}\\ \text{IN THE DIRECTION}\\ \text{OF MOTION}\\ \text{(COMMONLY CALLED "T")}}} \underbrace{\|r'(t)\|\, dt}_{\text{VELOCITY} \cdot \Delta\text{TIME} = \Delta d}$$

SO THIS FITS OUR IDEA OF WHAT WORK IS. LET'S DO SOME EXAMPLES:

EX: FIND THE WORK DONE BY $F = (x-y, x+y)$ ALONG THE PATH $r(t) = (\underset{x}{\underset{\parallel}{t}}, \underset{y}{\underset{\parallel}{\cos t}})$ FROM $t = 0$ TO π

$$W = \int F \cdot dr = \int F \cdot r'(t)\, dt$$

$$= \int_0^{\pi} (t - \cos t,\ t + \cos t) \cdot (1, -\sin t)\, dt$$

$$= \int t - \cos t - t\sin t - \cos t \sin t\, dt$$

$$= \left[\frac{1}{2}t^2 - \sin t\right]\Big|_0^{\pi} - \int t \sin t\, dt - \int \cos t \sin t\, dt$$

$u = \sin t$
$du = \cos t\, dt$

$$= \frac{\pi^2}{2} - \int t \sin t\, dt - \underbrace{\left[\frac{1}{2}\sin^2 t\right]\Big|_0^{\pi}}_{=0}$$

$$= \frac{\pi^2}{2} - \int t \sin t \, dt$$

$\uparrow$ INT. BY PARTS $\longrightarrow \int u \, dv = uv - \int v \, du$

$u = t \qquad dv = \sin t \, dt$

$du = dt \qquad v = -\cos t$

$$= \frac{\pi^2}{2} - \left[-t\cos t + \int \cos t \, dt \right]$$

$$= \frac{\pi^2}{2} - \left[-t\cos t + \sin t \right] \Big|_0^{\pi}$$

$$= \frac{\pi^2}{2} - [\pi]$$

$$= \frac{\pi^2}{2} - \pi$$

NOTE: ALL OF THESE SAME IDEAS AND CALCULATIONS HOLD FOR FOR LINE INTEGRALS IN $\mathbb{R}^3$, i.e. OF THE FORM $\int f\,dx + g\,dy + h\,dz$

EX: FIND THE WORK DONE BY A FORCE $F = (x, y, z^2)$ ALONG THE LINE $r(t) = (t+3, t-1, -t)$ FROM $t = 0$ TO 2.

$$W = \int F \cdot dr = \int F \cdot r'(t) \, dt$$

$$= \int_0^2 (t+3, t-1, t^2) \cdot (1, 1, -1) \, dt$$

$$= \int t + 3 + t - 1 - t^2 \, dt$$

$$= \int -t^2 + 2t + 2 \, dt$$

$$= \left[-\frac{1}{3}t^3 + t^2 + 2t \right]_0^2$$

$$= -\frac{8}{3} + 8$$

EX: COMPUTE $\int x^2\,ds$ ALONG THE CURVE $r(t) = (\cos t, \sin t, t)$ FROM $t = 0$ TO π

HERE $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\;dt$

$= \sqrt{(-\cos t)^2 + (\sin t)^2 + 1}\;dt$

$= \sqrt{2}\,dt$

$\int x^2\,ds = \int \cos^2 t\;\sqrt{2}\,dt$

$= \sqrt{2}\int \frac{1}{2}(1 + \cos 2t)\,dt$

$= \frac{1}{\sqrt{2}}\left[t + \frac{1}{2}\sin 2t\right]\Big|_0^{\pi}$

$= \frac{\pi}{\sqrt{2}}$

TRIG IDENTITIES YOU MUST KNOW: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

WHEN INTEGRATING AROUND A CLOSED LOOP, THE NOTATION $\oint P\,dx + Q\,dy + R\,dz$ IS OFTEN USED. A CLOSED LOOP FORCE INTEGRAL IS OFTEN CALLED THE CIRCULATION AROUND THE CURVE, SINCE IT MEASURES HOW MUCH F FLOWS IN THE DIRECTION OF YOUR LOOP.

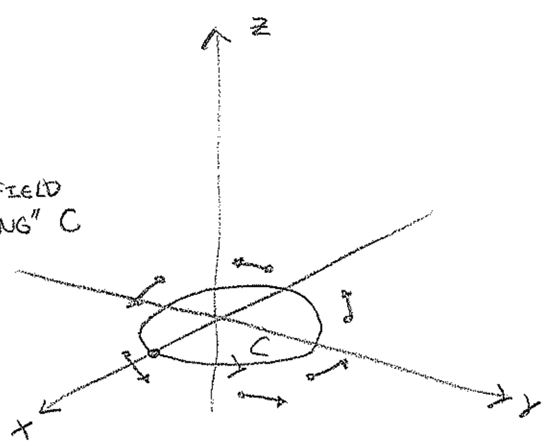
EX: $F = (-y, x, z)$. $r(t) = (\cos t, \sin t, 0)$. COMPUTE THE CIRCULATION $\oint F \cdot dr$

$\oint F \cdot dr = \int_0^{\pi} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt$

$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt$

$= \int_0^{2\pi} dt$

$= 2\pi > 0$ SO OUR FIELD "FLOWS ALONG" C



RMK: IF YOU TRAVERSE YOUR PATH IN YOUR LINE INTEGRAL THE IN OPPOSITE DIRECTION, YOU CHANGE YOUR INTEGRAL BY A SIGN.

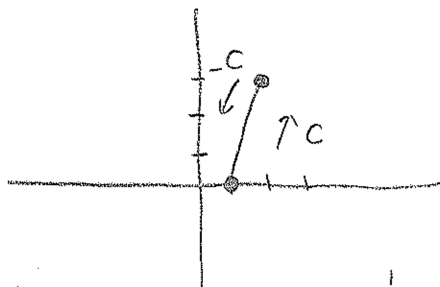
MUCH LIKE HOW $\int_a^b f\,dx = -\int_b^a f\,dx$ (IT ACTUALLY IS A RESULT OF THIS)

FOR EXAMPLE SUPPOSE OUR CURVE C IS THE LINE FROM $(1,0)$ TO $(2,3)$ GIVEN BY

$$r(t) = (t+1,\ 3t) \quad t = 0 \text{ TO } 1$$

WE DENOTE BY $-C$ AS THE PATH IN THE OPPOSITE DIRECTION GIVEN BY.

$$\tilde{r}(t) = (2-t,\ 3-3t)$$



THEN $\int_C P\,dx + Q\,dy = \int_0^1 P(dt) + Q(3dt) = \int_0^1 (P+3Q)\,dt$ $\begin{pmatrix} x = t+1 & y = 3t \\ dx = dt & dy = 3dt \end{pmatrix}$

AND $\int_{-C} P\,dx + Q\,dy = \int_0^1 P(-dt) + Q(-3dt)$ $\begin{pmatrix} x = 2-t & y = 3-3t \\ dx = -dt & dy = -3dt \end{pmatrix}$

$$= -\int (P+3Q)\,dt$$

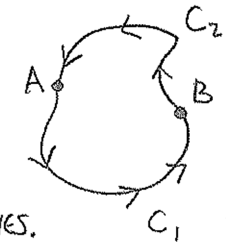
$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

ONE REMARK THAT IS IMPORTANT:

F HAS ALL LINE INTEGRALS $\int F \cdot dr$ BEING PATH INDEPENDENT $\overset{\text{IS EQUIVALENT TO}}{\Longleftrightarrow}$ $\oint F \cdot dr = 0$ AROUND ALL CLOSED LOOPS

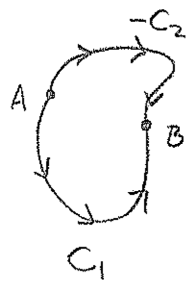
LET'S PROVE BOTH IMPLICATIONS:

$\Longrightarrow$ FIX SOME LOOP:



PICK TWO POINTS ON IT AND LET C_1, C_2 BE THE LABELLED CURVES.

THEN $\oint F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$



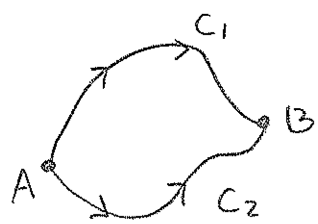
$$= \int_{C_1} F \cdot dr - \int_{-C_2} F \cdot dr = 0$$

C_1 & $-C_2$ ARE BOTH PATHS FROM A TO B, SO BY PATH INDEPENDENCE THEY ARE EQUAL!

THUS $\oint F \cdot dr = 0$ AROUND ANY LOOP

$\Longleftarrow$ PICK TWO POINTS A & B AND TWO PATHS C_1 AND C_2 FROM A TO B. WE WANT

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$



JUST VIEW THIS AS A CLOSED LOOP GOING ALONG C_1 & THEN BACKWARDS ON C_2! THIS INTEGRAL IS ZERO:

MOVE TO LEFT SIDE OF FAR LEFT EQ.

$$0 = \oint F \cdot dr = \int_{C_1} F \cdot dr + \int_{-C_2} F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr$$

SO

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

IN PARTICULAR:

$F = \nabla \phi \overset{\text{EQUIV}}{\Longleftrightarrow} F$ PATH INDEPENDENT $\overset{\text{EQUIV}}{\Longleftrightarrow} \oint F \cdot dr = 0$ OVER ALL CLOSED LOOPS

NOTICE THAT IF $F = (P, Q, R)$ THEN

$$\int F \cdot dr = \int (P,Q,R) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$= \int \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right] dt \quad \star$$

$$= \int P\,dx + Q\,dy + R\,dz$$

IS THE GENERAL FORM OF A LINE INTEGRAL

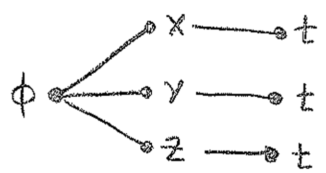
LIKEWISE (SAME STEPS BUT BACKWARDS) WE CAN THINK OF ANY LINE INTEGRAL AS A WORK INTEGRAL. WHEN THE FIELD F LOOKS LIKE THE GRADIENT $\nabla\phi$ OF SOME FUNCTION ϕ, SOMETHING HAPPENS:

$$F = \nabla\phi = \left(\overset{P}{\overset{\shortparallel}{\frac{\partial\phi}{\partial x}}}, \overset{Q}{\overset{\shortparallel}{\frac{\partial\phi}{\partial y}}}, \overset{R}{\overset{\shortparallel}{\frac{\partial\phi}{\partial z}}} \right)$$

SUPPOSE WE DO A LINE INTEGRAL FROM $\overset{\text{AT } t=t_1}{\overset{\uparrow}{(a_1, a_2, a_3)}}$ TO $\overset{\text{AT } t=t_2}{\overset{\uparrow}{(b_1, b_2, b_3)}}$ ALONG A CURVE C:

$$\int F \cdot dr = \int_{t_1}^{t_2} \underbrace{\left[\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \right]}_{= \frac{d\phi}{dt}} dt \quad \text{BY } \star$$

OUR "TREE DIAGRAM"



$$= \int \frac{d\phi}{dt}\, dt$$

FUNDAMENTAL THEOREM OF CALCULUS!

$$= \phi(t_2) - \phi(t_1)$$

$$= \phi(b_1, b_2, b_3) - \phi(a_1, a_2, a_3)$$

IN OTHER WORDS, THE VALUE OF OUR LINE INTEGRAL ONLY DEPENDED ON THE STARTING & ENDING POINTS OF OUR PATH. THIS IS CALLED PATH INDEPENDENT. IT OCCURS EXACTLY WHEN $F = \nabla\phi$ FOR SOME FUNCTION ϕ, CALLED THE POTENTIAL FUNCTION.

THERE ARE TWO REASONS WHY YOU MIGHT WANT TO USE POTENTIAL FUNCTIONS.

1. UGLY PATH

EX: $F = (y, x, 1)$ AND $r(t) = \left(e^{\cos t}, \cos(e^t), t\right)$ $t = 0$ TO π

FIND $\int F \cdot dr$

IF WE TRY TO DO IT NORMALLY WE GET:

$$\int F \cdot dr = \int_0^{\pi} \left(\cos(e^t), e^{\cos t}, 1\right) \cdot \left((-\sin t) e^{\cos t}, -\sin(e^t) e^t, 1\right) dt$$

$$= \int_0^{\pi} -\sin t \cos(e^t) e^{\cos t} - \sin(e^t) e^{t+\cos t} + 1 \; dt$$

THIS IS AWFUL.

BUT IF $F = \nabla\phi = (\phi_x, \phi_y, \phi_z)$ FOR SOME ϕ, THIS WOULD BE EASY.

SO, WE TRY TO RECOVER THE FUNCTION ϕ BASED ON ITS PARTIAL DERIVATIVES.

SINCE TAKING $\frac{\partial}{\partial x}$ OF ϕ WILL WIPE OUT ANY TERMS WITHOUT X'S (i.e. TERMS CONTAINING Y'S AND Z'S ONLY), IF WE INTEGRATE ϕ_x WITH RESPECT TO X TO TRY AND FIND ϕ WE MAY BE MISSING SOME OTHER TERMS INVOLVING ONLY Y & Z.

$$\phi = \int \frac{\partial\phi}{\partial x}\, dx = \int y\, dx = xy + C_1(y,z)$$

$$\phi = \int \frac{\partial\phi}{\partial y}\, dy = \int x\, dy = xy + C_2(x,z)$$

$$\phi = \int \frac{\partial\phi}{\partial z}\, dz = \int dz = z + C_3(x,y)$$

SO ϕ SHOULD HAVE BOTH AN XY AND Z TERM, SO LET $\phi = xy + z$ (CHECK $\nabla\phi = F$)

SO $\int F \cdot dr = \int \nabla\phi \cdot dr = \phi(r(\pi)) - \phi(r(0))$

$r(\pi) = \left(e^{-1}, \cos(e^{\pi}), \pi\right)$

$r(0) = \left(e, \cos(1), 0\right)$

$$= \phi\left(e^{-1}, \cos(e^{\pi}), \pi\right) - \phi(e, \cos 1, 0)$$

$$= \left[e^{-1}\cos(e^{\pi}) + \pi\right] - \left[e \cos 1\right]$$

② UGLY F

EX: $F = \left(y - z\sin(xz),\ x + e^z,\ ye^z - x\sin(xz)\right)$

$r(t) = (\sin t, \cos t, t)$ $t = 0$ TO $\frac{\pi}{2}$

COMPUTE $\int F \cdot dr$

AGAIN THIS WILL BE AWFUL IF YOU DO IT DIRECTLY. LET'S HOPE THAT $F = \nabla\phi$.

IF SO $F = (\phi_x, \phi_y, \phi_z)$ AND THUS

$$\phi = \int \frac{\partial\phi}{\partial x}\,dx = \int y - z\sin(xz)\,dx = xy + \cos(xz) + C_1(y,z)$$

$$\phi = \int \frac{\partial\phi}{\partial y}\,dy = \int x + e^z\,dy = xy + ye^z + C_2(x,z)$$

$$\phi = \int \frac{\partial\phi}{\partial z}\,dz = \int ye^z - x\sin(xz)\,dz = ye^z + \cos(xz) + C_3(x,y)$$

SO ϕ NEED AN xy TERM, A ye^z TERM, & A $\cos(xz)$ TERM:

$\phi = xy + ye^z + \cos(xz)$ AND CHECK THAT $\nabla\phi = F$

$$\int F \cdot dr = \int \nabla\phi \cdot dr = \phi\left(r\left(\tfrac{\pi}{2}\right)\right) - \phi(r(0))$$

$$= \phi\left(1, 0, \tfrac{\pi}{2}\right) - \phi(0,1,0)$$

$$= \cancel{\cos\left(\tfrac{\pi}{2}\right)} - [0 + 1 + 1]$$

$$= -2$$

NOTE THAT F BEING OF THE FORM $\nabla\phi$ IS A VERY SPECIAL PROPERTY. THIS FUNCTION NEED NOT EXIST. IF IT DOES, F IS CALLED CONSERVATIVE. SO WHAT HAPPENS IF F IS NOT CONSERVATIVE? WHERE DOES OUR METHOD BREAK DOWN?

EX: IS $F = (x,x,x)$ CONSERVATIVE? (SO $\phi_x = \phi_y = \phi_z = x$)

IF SO, $\phi = \int \phi_x\,dx = \int x\,dx = \frac{1}{2}x^2 + C_1(y,z)$ → i.e. NO OTHER TERM IN ϕ HAS AN x IN IT (OTHER THAN $\frac{1}{2}x^2$)

$\phi = \int \phi_y\,dy = \int x\,dy = \boxed{xy} + C_2(x,z)$

→ ϕ MUST HAVE AN xy TERM, THIS CONTRADICTS

THERE IS ANOTHER WAY TO TEST THAT A FIELD IS CONSERVATIVE.

IF $F = \nabla\phi = (P, Q, R)$ THEN $P = \phi_x$, $Q = \phi_y$, $R = \phi_z$

UNDER MILD ASSUMPTIONS THE MIXED PARTIALS ARE EQUAL:

$$\phi_{xy} = \phi_{yx}, \quad \phi_{xz} = \phi_{zx}, \quad \phi_{yz} = \phi_{zy}$$

WHICH IN TERMS OF F IMPLIES:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

IN $\mathbb{R}^3$ THIS IS THE SAME AS $\text{CURL } F = 0$ (LOOK AT DEFINITION OF CURL)

SO $F = \nabla\phi$ IMPLIES $\text{CURL } F = 0$

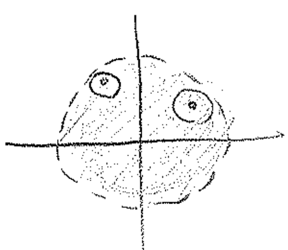
THE OTHER IMPLICATION IS TRUE <u>SOMETIMES</u>

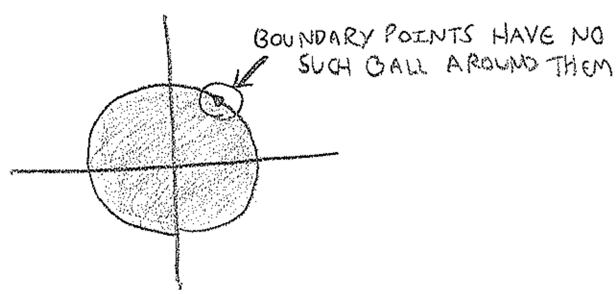
<u>THM</u>: IF F IS A VECTOR FIELD ON AN OPEN, SIMPLY CONNECTED DOMAIN IN $\mathbb{R}^3$ SUCH THAT $\text{CURL } F = 0$, THEN $F = \nabla\phi$ FOR SOME FUNCTION ϕ.

OPEN MEANS THAT EVERY POINT IN THE DOMAIN HAS A SMALL BALL AROUND IT THAT IS ALSO CONTAINED IN THE DOMAIN.

$\{(x,y) \mid x^2+y^2 < 1\}$ IS OPEN

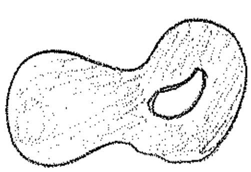
$\{(x,y) \mid x^2+y^2 \leq 1\}$ IS NOT OPEN



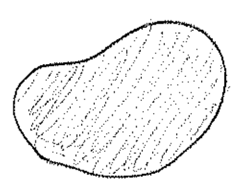


SIMPLY CONNECTED MEANS THAT THERE ARE "NO HOLES" IN THE DOMAIN:

NOT SIMPLY CONNECTED



SIMPLY CONNECTED



IN $\mathbb{R}^2$ THIS THEOREM JUST BECOMES:

<u>THM</u> IF F IS A VECTOR FIELD ON AN OPEN, SIMPLY CONNECTED DOMAIN IN $\mathbb{R}^2$ s.t. (ASSUMING $F=(P,Q)$) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, THEN $F = \nabla\phi$ FOR SOME FUNCTION ϕ.

THIS IS BECAUSE IF $F = (\phi_x, \phi_y)$ EQUALITY OF MIXED PARTIALS ONLY GETS US 1 EQUATION

$$\phi_{xy} = \phi_{yx} \longrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

IN PARTICULAR, $\mathbb{R}^2$ & $\mathbb{R}^3$ ARE SIMPLY CONNECTED AND OPEN. SO WE CAN USE THE EQUALITY OF MIXED PARTIALS OF A GLOBALLY DEFINED VECTOR FIELD TO DEDUCE THAT OUR F IS CONSERVATIVE. WE FINISH WITH AN EXAMPLE OF WHEN OUR DOMAIN IS NOT SIMPLY CONNECTED:

EX: $F = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q}\Big)$ ON $\mathbb{R}^2$ - ORIGIN (NOT SIMPLY CONNECTED)

IT SATISFIES $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ SO WE HOPE $F = \nabla\phi$ FOR SOME FUNCTION.

IN THIS CASE, $\oint F\cdot dr = 0$ AROUND ANY CLOSED LOOP. SO LET'S INTEGRATE AROUND THE UNIT CIRCLE: $r(t) = (\cos t, \sin t)$

$$\oint F\cdot dr = \int_0^{2\pi}\left(\frac{-\sin t}{\cancel{\cos^2 t + \sin^2 t}}, \frac{\cos t}{\cancel{\cos^2 t + \sin^2 t}}\right)\cdot(-\sin t, \cos t)\, dt$$

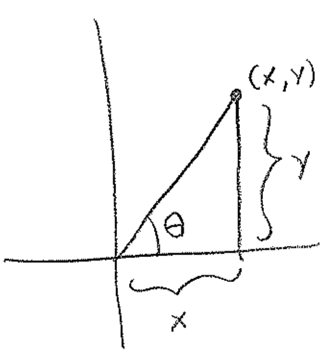
$$= \int \sin^2 t + \cos^2 t \, dt$$

$$= \int dt = 2\pi \neq 0$$ SO $F \neq \nabla\phi$ FOR <u>ANY</u> FUNCTION ϕ

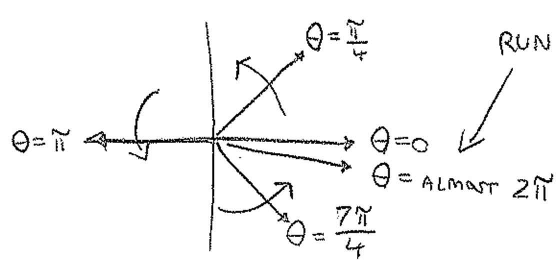
BUT

IF WE LET $\phi = \arctan\left(\frac{x}{y}\right)$ THEN $\phi_x = \frac{y}{x^2+y^2}$, $\phi_y = \frac{-x}{x^2+y^2}$ THEN $\nabla\phi = -F$

WHY DOES THIS NOT WORK? AT A POINT (x,y), $\arctan\left(\frac{x}{y}\right) = \theta$ (ANGLE IN PICTURE BELOW)



WE CAN'T DEFINE THIS ANGLE FUNCTION ON ALL OF $\mathbb{R}^2 - \{(0,0)\}$ WITHOUT IT BEING DISCONTINUOUS!!



RUN INTO CONTINUITY PROBLEMS!