LAST TIME: WE TALKED ABOUT LINE INTEGRALS. WE SAW HOW ANY LINE INTEGRAL WHEN WRITTEN IN THE FORM

$$\int P\,dx + Q\,dy + R\,dz \quad (\text{IN } \mathbb{R}^3) \quad \text{OR} \quad \int P\,dx + Q\,dy \quad (\text{IN } \mathbb{R}^2)$$

CAN BE THOUGHT OF AS A WORK INTEGRAL $\int F \cdot dr$ WITH FORCE FIELD

$$F = (P, Q, R) \text{ IN } \mathbb{R}^3 \qquad F = (P, Q) \text{ IN } \mathbb{R}^2$$

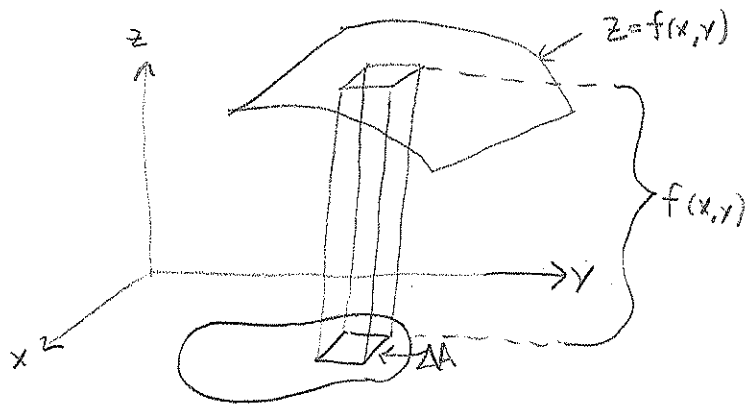
WE ALSO SAW THAT THIS LINE INTEGRAL BEING INDEPENDENT OF PATH WAS EQUIVALENT TO $F = \nabla\phi$ (GRADIENT OF ϕ) FOR SOME FUNCTION ϕ, CALLED THE POTENTIAL FUNCTION. IN THIS CASE $\int F \cdot dr = \phi(B) - \phi(A)$ WHERE OUR PATH STARTS AT A AND ENDS AT B.

DOUBLE INTEGRALS

WE WOULD LIKE TO INTEGRATE FUNCTIONS OVER 2-DIMENSIONAL DOMAINS IN $\mathbb{R}^2$.

WE WILL WRITE THIS AS: $\iint f(x,y)\, \underbrace{dA}_{\text{SMALL PIECE OF AREA}}$

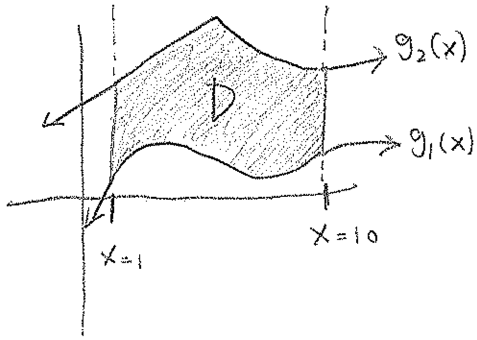
IF WE PICTURE THE SURFACE $z = f(x,y)$ IN $\mathbb{R}^3$ THIS WOULD REPRESENT THE VOLUME BETWEEN THIS SURFACE AND THE XY-PLANE:



IF $f = 1$, $\iint f\,dA = \iint dA$ = AREA OF OUR DOMAIN IN $\mathbb{R}^2$

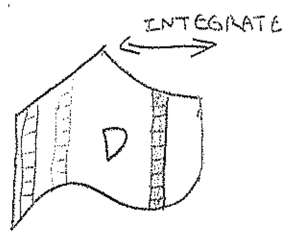
TO COMPUTE THESE INTEGRALS WE USE <u>ITERATED INTEGRALS</u>. THIS IS JUST A METHOD OF INTEGRATING ONE VARIABLE AT A TIME.

FOR EXAMPLE, SUPPOSE WE WANTED TO COMPUTE $\iint f\, dA$ IN THE DOMAIN D:



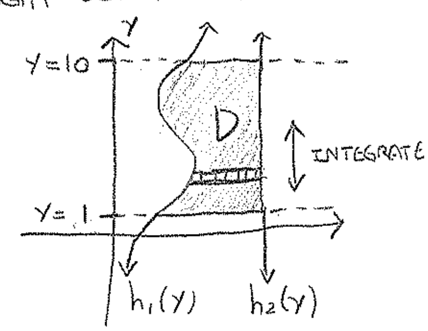
THEN $\displaystyle\int_1^{10}\left[\int_{g_1(x)}^{g_2(x)} f\, dy\right] dx = \iint_D f\, dA$

FIRST WE INTEGRATE ALONG THE VERTICAL STRIPS FROM $g_1(x)$ TO $g_2(x)$ FOR ANY X:



THEN WE INTEGRATE THESE STRIPS FROM LEFT TO RIGHT TO FILL OUT OUR REGION D.

SIMILARLY WE MIGHT USE FUNCTIONS IN Y IF OUR DOMAIN IS SHAPED A LITTLE DIFFERENT:



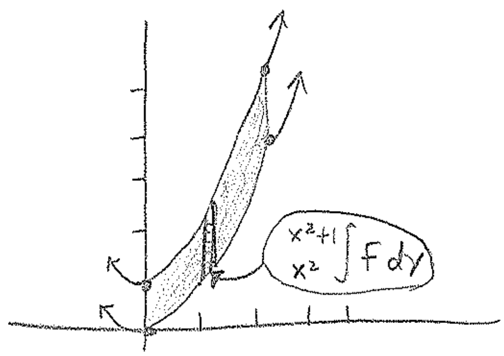
THEN $\displaystyle\int_1^{10}\left[\int_{h_1(y)}^{h_2(y)} f\, dx\right] dy = \iint f\, dA$

WE CAN COMPUTE THESE INTEGRALS IN <u>EITHER WAY</u> BUT SOMETIMES ONE ORDER OF INTEGRATION WILL BE EASIER THAN THE OTHER SINCE ON MAY REQUIRE YOU TO SPLIT UP THE DOMAIN INTO PIECES. NOTICE THAT IN BOTH OF OUR PICTURES ABOVE THAT SWITCHING THE ORDER OF INTEGRATION WOULD REQUIRE THIS SPLITTING OF OUR DOMAIN.

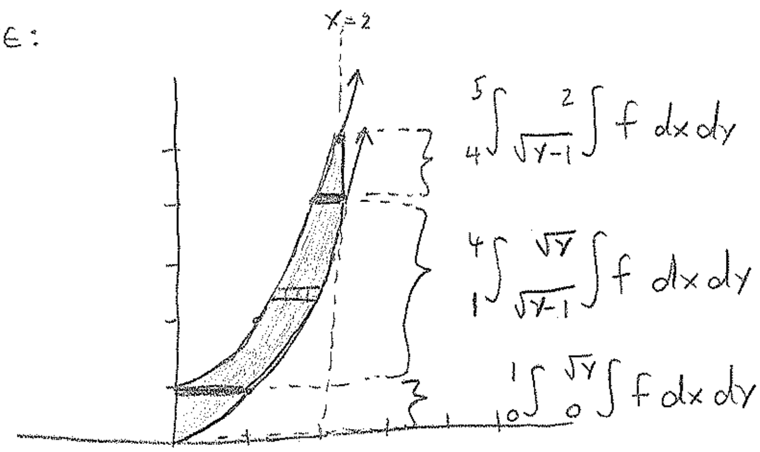
$\overbrace{x+y}^{f=x+y}$

EX: COMPUTE $\iint x+y \, dA$ ON THE AREA BETWEEN $y=x^2$, $y=x^2+1$, $x=0$, $x=2$.

FIRST, DRAW THE DOMAIN SO YOU CAN DECIDE THE ORDER OF INTEGRATION & THE LIMITS!



HERE IT IS APPARENT THAT WE WANT TO USE VERTICAL STRIPS (i.e. $\int$ WITH RESPECT TO y FIRST) SINCE THE OUR LIMITS ARE $y=x^2$ TO $y=x^2+1$ IN ONE INTEGRAL. OTHERWISE, OUR PICTURE LOOKS LIKE:



IN OTHER WORDS WE NEED TO COMPUTE 3 INTEGRALS!

$$\text{So } \iint f \, dA = \int_0^2 \int_{x^2}^{x^2+1} x+y \, dy\, dx$$

$$= \int_0^2 \left[xy + \tfrac{1}{2}y^2\right]\Big|_{x^2}^{x^2+1} dx$$

$$= \int_0^2 \left[\cancel{x^3} + x + \tfrac{1}{2}(x^2+1)^2 - \left[\cancel{x^3} + \tfrac{1}{2}x^4\right]\right] dx$$

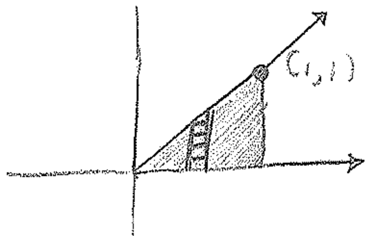
$$= \int_0^2 x + \tfrac{1}{2}(\cancel{x^4} + 2x^2 + 1) - \cancel{\tfrac{1}{2}x^4} \, dx$$

$$= \int_0^2 x^2 + x + \tfrac{1}{2} \, dx$$

$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{2}x\right]\Big|_0^2 = \frac{8}{3} + 3$$

ANOTHER FACTOR IN CHOOSING THE ORDER OF INTEGRATION IS f ITSELF:

EX: $\iint e^{x^2}\, dA$ IN THE REGION BETWEEN $x=0$, $x=1$, $y=x$



SINCE WE CAN'T DO $\int e^{x^2} dx$ WE CHOOSE TO INTEGRATE y FIRST:

$$\iint e^{x^2}\, dA = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left[y e^{x^2} \right] \Big|_0^x \, dx$$

$$= \int_0^1 x e^{x^2}\, dx \qquad u = x^2,\ du = 2x\, dx$$

$$= \frac{1}{2} \int e^u\, du$$

$$= \frac{1}{2} \left[e^{x^2} \right] \Big|_0^1$$

$$= \frac{1}{2}(e-1)$$

POLAR COORDINATES (r, θ)

WHEN INTEGRATING OVER CIRCULAR OR ELLIPTICAL DOMAINS IT'S USUALLY EASIER TO DO IN POLAR COORDINATES. OUR "AREA ELEMENT" dA, WHICH IN CARTESIAN COORDINATES IS $dA = dx\,dy$ BECOMES:

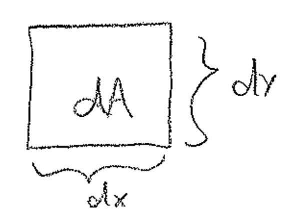
$$dA = r\, dr\, d\theta$$

RECALL:

$x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

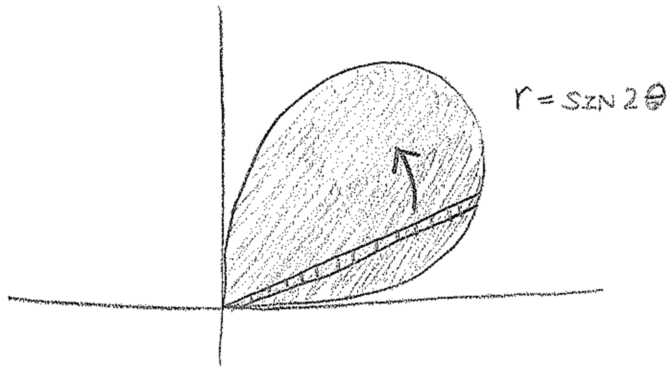


CARTESIAN

POLAR

AGAIN, WE USE ITERATED INTEGRALS TO COMPUTE THESE.

EX: COMPUTE THE VOLUME BELOW THE SURFACE $Z = X^2 + Y^2$ AND ABOVE THE REGION GIVEN BY $r = \sin 2\theta$ IN THE FIRST QUADRANT OF THE XY-PLANE.



THE VOLUME IS:

$$V = \iint x^2 + y^2 \, dA \quad \text{USE POLAR COORDINATES}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r^2 \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\left[r^3\right]\Big|_0^{\sin 2\theta} \, d\theta$$

$$= \frac{1}{3} \int \sin^3 2\theta \, d\theta$$

$$= \frac{1}{3} \int \sin 2\theta \left(1 - \cos^2 2\theta\right) d\theta \longrightarrow \begin{array}{l} \text{LET } u = \cos 2\theta \\ du = -2 \sin 2\theta \, d\theta \end{array}$$

$$= -\frac{1}{6} \int 1 - u^2 \, du$$

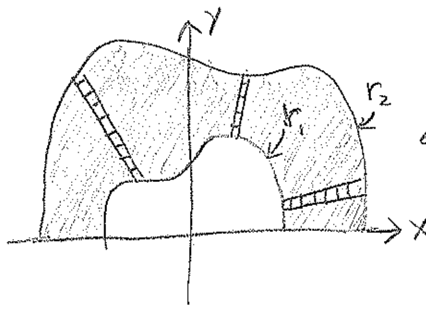
$$= -\frac{1}{6} \left[\cos 2\theta - \frac{1}{3}\cos^3 2\theta\right]\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{6}\left[0 - \left(1 - \frac{1}{3}\right)\right]$$

$$= -\frac{1}{6}\left(-\frac{2}{3}\right)$$

$$= \frac{1}{9}$$

WE TYPICALLY INTEGRATE WITH RESPECT TO r FIRST AND VIEW OUR ITERATED INTEGRAL AS FIRST INTEGRATING RADIAL PATHS OUTWARD FROM THE ORIGIN AND THEN INTEGRATING THESE BY θ:



HERE $\iint f \, dA = \int_0^{\pi} \int_{r_1}^{r_2} f \, dr \, d\theta$

AGAIN, SOMETIMES YOU MAY NEED TO BREAK UP YOUR DOMAIN:

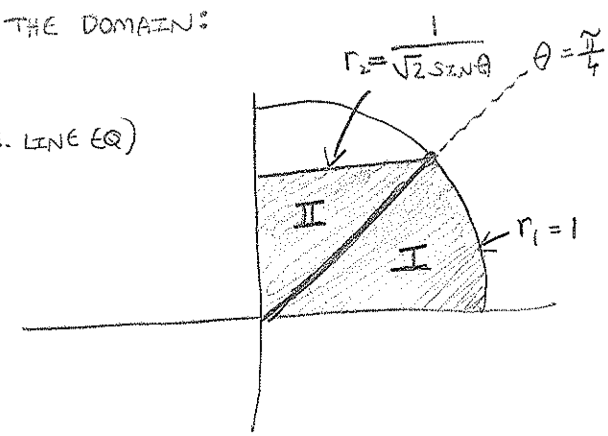
EX: COMPUTE $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \int_0^{\sqrt{1-y^2}} \frac{y^2}{x^2+y^2} \, dx \, dy$

HERE WE WANT TO CHANGE TO POLAR COORDINATES SINCE IT WILL SIMPLIFY THE INTEGRAND.

ALSO NOTE THE SHAPE OF THE DOMAIN:

$y = r\sin\theta = \frac{1}{\sqrt{2}}$ (HORIZ. LINE EQ)

$r = \dfrac{1}{\sqrt{2}\sin\theta}$



WE NEED TO SPLIT UP OUR INTEGRAL INTO 2 PIECES, AS SHOWN ABOVE.

$$\iint \frac{y^2}{x^2+y^2} \, dx \, dy = \underbrace{\int_0^{\frac{\pi}{4}} \int_0^{1} \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}} \, r \, dr \, d\theta}_{\text{I}} + \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sqrt{2}\sin\theta}} \frac{r^2\sin^2\theta}{r^2} \, r \, dr \, d\theta}_{\text{II}}$$

$$\text{I} = \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 \Big|_0^1 \sin^2\theta \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{2}(1 - \cos 2\theta) \, d\theta \quad \text{(TRIG IDENTITY)}$$

$$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{1}{16}\left[\pi - 2\right]$$

$$(\mathrm{II}) = \int \frac{1}{2} r^2 \Big|_0^{\frac{1}{\sqrt{2}\sin\theta}} \sin^2\theta \, d\theta$$

$$= \frac{1}{4} \int \frac{1}{\sin^2\theta} \sin^2\theta \, d\theta$$

$$= \frac{1}{4} \int d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{16}$$

SO FINAL ANSWER IS $\frac{1}{16}\left[2\pi - 2\right] = \frac{1}{8}\left[\pi - 1\right]$

GREEN'S THM (ONLY FOR $\mathbb{R}^2$)

GREENS THEOREM HELPS CONVERT LINE INTEGRALS AROUND A CLOSED LOOP (ONLY!!) TO AN AREA INTEGRAL INSIDE ITS INTERIOR:

THM IF $P, Q, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x}$ ARE ALL CONTINUOUS FUNCTIONS ON SOME REGION R ENCLOSED BY A CURVE C WHICH DOES NOT CROSS ITSELF, THEN:

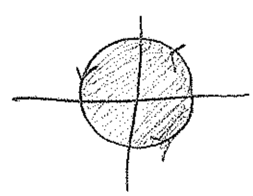
$$\oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy$$ (ASSUME WE GO IN ↺ DIRECTION ON C)

THIS IS ACTUALLY A SPECIAL CASE OF STOKES' THM. THE IDEA IS THAT $\oint P\,dx + Q\,dy = \oint F \cdot dr$ IS A CIRCULATION INTEGRAL AROUND SOME CURVE C AND $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ MEASURES THE CIRCLUATION AROUND A TINY BOX, AND ADJACENT CIRCULATIONS CANCEL.

GREEN'S THM CAN BE VERY USEFUL TO GET AROUND DOING TOUGH LINE INTEGRALS:

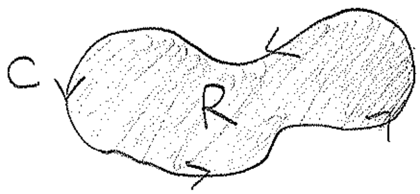
EX: COMPUTE THE LINE INTEGRAL AROUND THE UNIT CIRCLE ↺ THIS WAY:

$$\oint \underbrace{\left(-y + \ln(|\cos x|)\right)}_{P} dx + \underbrace{\left(x + e^{\cos y}\right)}_{Q} dy$$



$$= \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \iint 1 - (-1)\, dx\, dy = 2 \iint dx\, dy = 2\pi$$

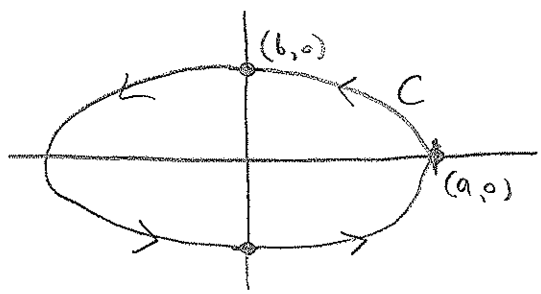
WE CAN USE GREEN'S THM TO COMPUTE AREAS OF DOMAINS:



$$\frac{1}{2}\oint_C \underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy = \frac{1}{2}\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \frac{1}{2}\iint_R 1+1\,dx\,dy = \iint_R dx\,dy = \text{AREA OF } R$$

GREEN'S

EX: COMPUTE THE AREA OF THE ELLIPSE: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

OUR CURVE IS: $r(t) = (\underset{x}{a\cos t}, \underset{y}{b\sin t})$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin t\,(-a\sin t) + a\cos t\,(b\cos t)\,dt$$

$$= \frac{1}{2}\int ab\left[\sin^2 t + \cos^2 t\right]dt$$

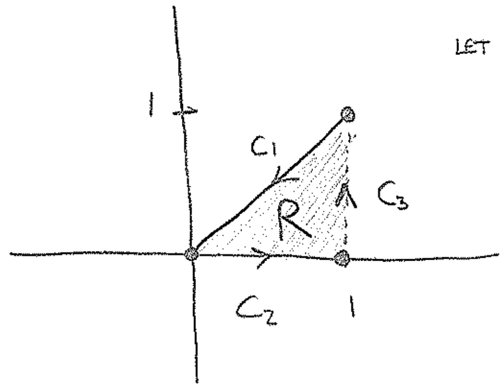
$$= \frac{ab}{2}\int dt = \pi ab$$

NOTE HOW MUCH SIMPLER THIS WAS THAN DOING $A = 2\int_{-a}^{a} b^2\sqrt{1-\frac{x^2}{a^2}}\,dx$

ANOTHER COMMON TRICK WHEN DOING A DIFFICULT LINE INTEGRAL ALONG A PATH WHICH IS NOT A CLOSED LOOP IS TO ADD ANOTHER PATH TO COMPLETE A LOOP SO GREEN'S THM CAN BE APPLIED:

EX: COMPUTE $\int_{C_1+C_2} (y^2 + e^{(x-1)^2})\,dx + xy\,dy$ WHERE C_1 AND C_2 ARE:

LET $F = (y^2 + e^{(x-1)^2}, xy)$



HERE DIRECTLY COMPUTING WILL RUN INTO PROBLEMS:

$$\int_{C_2} (y^2 + e^{(x-1)^2})\,dx + xy\,dy = \int_0^1 e^{(t-1)^2}\,dt \quad \text{IF } r(t) = (t, 0) \quad t = 0 \text{ TO } 1 \text{ IS OUR PATH.}$$

CAN'T DO!

SO WE COMPLETE OUR CURVES WITH C_3 TO FORM A LOOP. GREEN'S THM SAYS:

$$\underbrace{\int_{C_1+C_2} F\cdot dr}_{\text{WHAT WE WANT}} + \underbrace{\int_{C_3} F\cdot dr}_{\text{EASY TO COMPUTE?}} = \iint_R y - 2y\,dx\,dy = \underbrace{-\iint_R y\,dx\,dy}_{\text{EASY TO COMPUTE}}$$

LET'S SEE:

LET $r(t) = (1, t)$ $t = 0$ TO 1 FOR C_3

$$\int_{C_3} F\cdot dr = \int_0^1 t^2(\underset{dx}{0}) + t\,\underset{dy}{dt} = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$$

THUS:

$$\int_{C_1+C_2} F\cdot dr = -\frac{1}{2} - \iint y\,dx\,dy$$

$$= -\frac{1}{2} - \int_0^1 \int_0^x y\,dy\,dx$$

$$= -\frac{1}{2} - \int \frac{1}{2}x^2\,dx$$

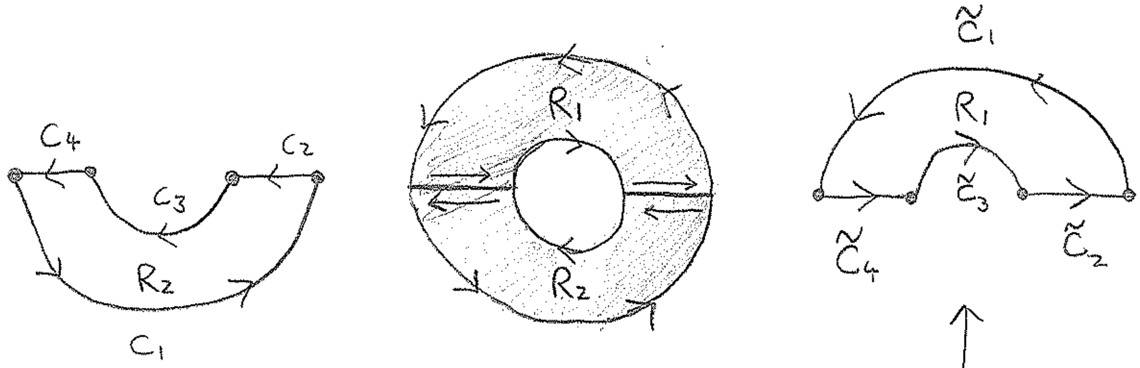
$$= -\frac{1}{2} - \frac{1}{6}x^3\Big|_0^1$$

$$= -\frac{1}{2} - \frac{1}{6}$$

$$= -\frac{2}{3}$$

INTEGRATING ON DOMAINS WITH HOLES

GREEN'S THEOREM CAN BE APPLIED TO DOMAINS WITH HOLES BY SPLITTING THE DOMAIN UP INTO PIECES GIVEN BY LOOP PATHS:



↑ GREEN'S:

$$\iint_{R_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \int_{C_1+C_2+C_3+C_4} F \cdot dr$$

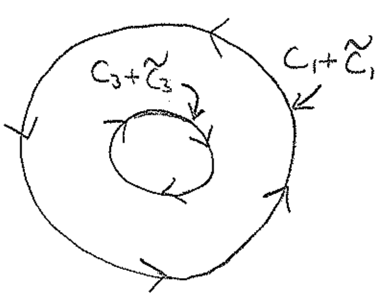
↑ GREEN'S:

$$\iint_{R_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \int_{\tilde{C}_1+\tilde{C}_2+\tilde{C}_3+\tilde{C}_4} F \cdot dr$$

BUT NOTICE THAT C_2 & $\tilde{C}_2$ (AND C_4 & $\tilde{C}_4$) ARE THE SAME PATH BUT IN OPPOSING DIRECTIONS SO IF WE ADD THESE TWO EQUATIONS THOSE INTEGRALS CANCEL, AND WE GET:

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \int_{C_1+C_3+\tilde{C}_1+\tilde{C}_3} F \cdot dr \longrightarrow$$

THIS CAN BE PICTURED:

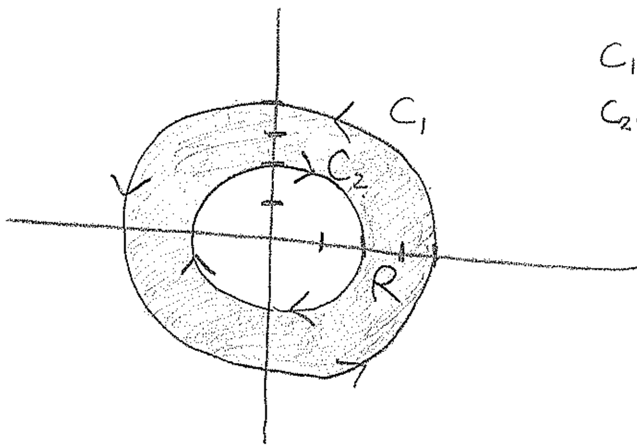


SO WE TRAVERSE THE PATHS WITH OUR REGION R TO OUR LEFT ON EACH PATH. THIS CAN BE APPLIED TO ANY # OF HOLES. NOTICE THAT IF $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ THEN THIS SAYS THAT THE INTEGRALS AROUND THE CIRCLES ARE EQUAL IF YOU TRAVERSE BOTH IN THE SAME DIRECTION, SINCE:

$$0 = \int_{C_1+\tilde{C}_1} F \cdot dr + \int_{C_3+\tilde{C}_3} F \cdot dr$$

MOVE OVER & SWITCH DIRECTION OF PATH

EX: COMPUTE $\oint_C -x^2y\,dx + xy^2\,dy$ WHERE C IS THE SUM OF THE PATHS C_1 & C_2:



C_1: CIRCLE RADIUS 4

C_2: CIRCLE RADIUS 2

GREEN'S THM:

$$\oint_C -x^2y\,dx + xy^2\,dy = \iint_R y^2 + x^2\,dx\,dy$$ CONVERT TO POLAR

$$= \int_0^{2\pi}\int_2^4 r^3\,dr\,d\theta$$

$$= \frac{1}{4}\int r^4\Big|_2^4\,d\theta$$

$$= 60\int d\theta$$

$$= 120\pi$$