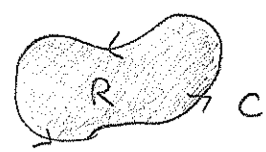
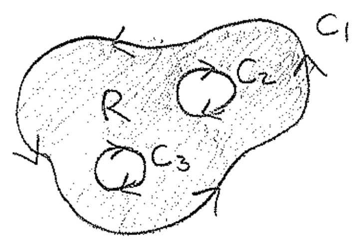
LAST TIME: WE TALKED ABOUT DOUBLE INTEGRALS, POLAR COORDINATE, AND MOST IMPORTANTLY GREEN'S THM:

$$\oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$



OR



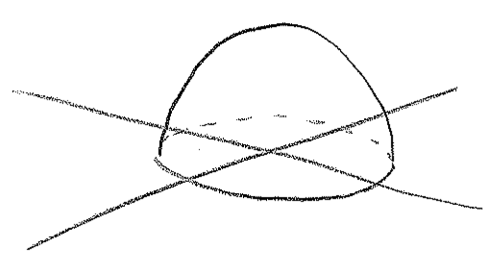
$C = C_1 + C_2 + C_3$

SURFACE INTEGRALS

WE USE SURFACE INTEGRALS TO COMPUTE SURFACE AREAS AND INTEGRATE FUNCTIONS ALONG SURFACES IN $\mathbb{R}^3$.

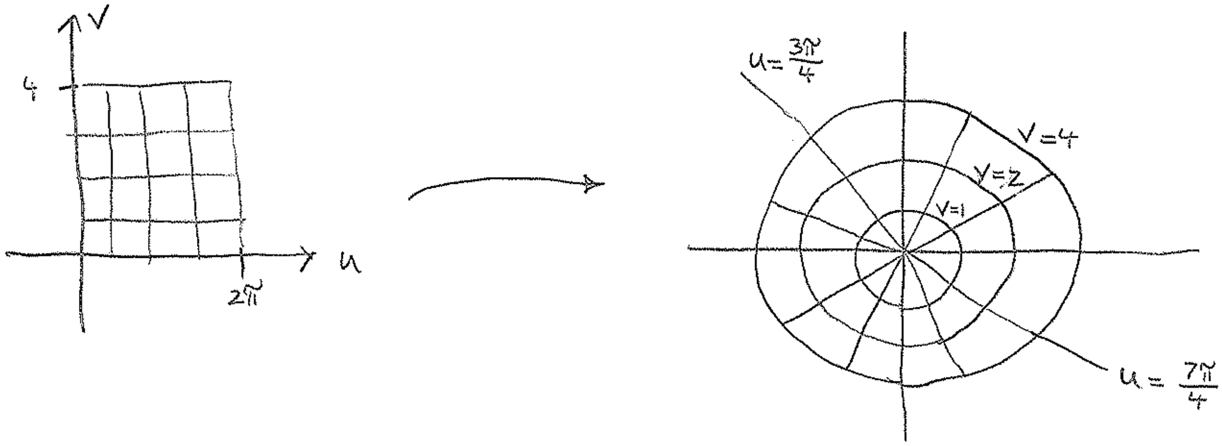
DEF: A PARAMETRIZED SURFACE IN $\mathbb{R}^3$ IS A FUNCTION $\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ WHOSE IMAGE IS A NICE SMOOTH SURFACE IN $\mathbb{R}^3$.

EX: IF $\sigma(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$ FOR (u,v) IN THE UNIT CIRCLE $\left(u^2+v^2 \leq 1\right)$ THEN THIS PARAMETRIZES THE UPPER HALF UNIT SPHERE:



EX: $\sigma(u,v) = (v\cos u,\ v\sin u,\ 0)$ FOR $0 \le u \le 2\pi,\ 0 \le v \le 4$

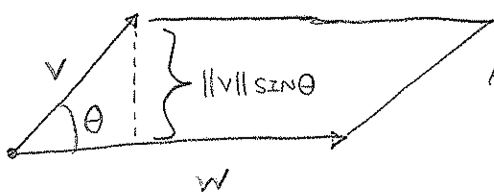
THIS PARAMETRIZES A DISC OF RADIUS 4 IN THE XY-PLANE:



NOTICE THAT HOLDING u OR V CONSTANT GIVES US A SET OF PATHS IN OUR SURFACE THAT SPLIT IT UP INTO BLOCKS. TO COMPUTE THE SURFACE AREA WE ESTIMATE THE AREA OF THESE "BLOCKS".

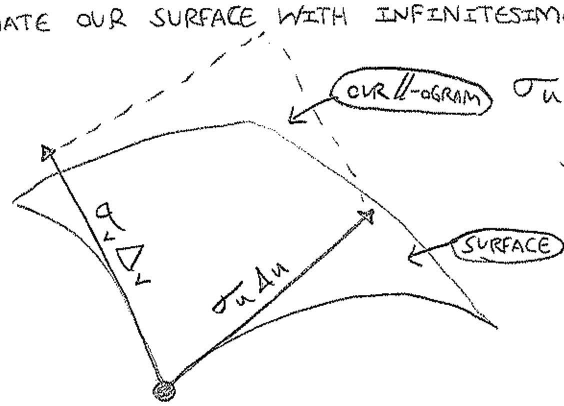
RECALL THAT FOR 2 VECTORS V & W:

$$\|V \times W\| = \|V\|\,\|W\| \sin\theta = \text{AREA OF}$$ 



$$\text{AREA} = bh = \|W\|\,\|V\| \sin\theta$$

NOW WE APPROXIMATE OUR SURFACE WITH INFINITESIMAL PARALLELOGRAMS:



σ_u DENOTES $\frac{\partial \sigma}{\partial u}$ (A VECTOR)

SO THAT $\sigma_u \Delta u$ APPROXIMATES HOW MUCH σ CHANGES IF WE CHANGE u.

AREA OF PIECE OF SURFACE $\approx$ AREA OF  $= \|\sigma_u \Delta u \times \sigma_v \Delta v\|$ PULL OUT SCALARS

$$= \|\sigma_u \times \sigma_v\|\,\Delta u\,\Delta v$$

SO TAKING A LIMIT AS OUR # OF 'S GOES TO ∞ WE GET:

$$\text{SURFACE AREA} = \iint \underbrace{\|\sigma_u \times \sigma_v\| \, du \, dv}_{\substack{\text{DENOTED } dS\text{, CALLED THE} \\ \text{SURFACE AREA ELEMENT}}} = \iint dS$$

NOTE: TYPICALLY OUR PARAMETERS WILL BE TWO OF X, Y, AND Z.

EX: COMPUTE THE SURFACE AREA OF THE HEMISPHERE $Z = \sqrt{4 - x^2 - y^2}$

WE USE $\sigma(x,y) = \left(x, y, \sqrt{4 - x^2 - y^2}\right)$ AS OUR PARAMETRIZATION

$$\sigma_x = \left(1, 0, \frac{-x}{\sqrt{4 - x^2 - y^2}}\right)$$

$$\sigma_y = \left(0, 1, \frac{-y}{\sqrt{4 - x^2 - y^2}}\right)$$

$$\sigma_x \times \sigma_y = \left(\frac{x}{\sqrt{4 - x^2 - y^2}}, \frac{y}{\sqrt{4 - x^2 - y^2}}, 1\right)$$

$$SA = \iint \|\sigma_x \times \sigma_y\| \, dx \, dy$$

$$= \iint \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} \, dx \, dy$$

$$= \iint \frac{2}{\sqrt{4 - x^2 - y^2}} \, dx \, dy$$

WHAT IS OUR DOMAIN? CIRCLE RADIUS 2

SO CONVERT TO POLAR

$$= \int_0^{2\pi} \int_0^2 \frac{2r}{\sqrt{4 - r^2}} \, dr \, d\theta \longrightarrow \begin{array}{l} u = 4 - r^2 \\ du = -2r\,dr \end{array}$$

$$= -\iint \frac{1}{\sqrt{u}} \, du \, d\theta$$

$$= -2 \int \sqrt{4 - r^2} \Big|_0^2 \, d\theta$$

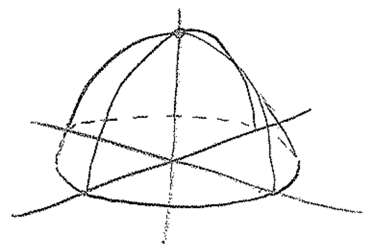
$$= -2 \int 0 - 2 \, d\theta = 4 \int_0^{2\pi} d\theta = 8\pi$$

WE CAN ALSO INTEGRATE FUNCTIONS ALONG OUR SURFACES:

$$\iint f \, dS$$

FOR EXAMPLE, f COULD BE A DENSITY FUNCTION SO THAT $M = \iint f \, dS$ IS THE MASS.

EX: COMPUTE $\iint x(4x^2+4y^2+1)^{\frac{1}{2}} \, dS$ ON THE SURFACE $z = 4 - x^2 - y^2$, $(z \geq 0)$.



HERE WE USE: $\sigma(x,y) = (x, y, 4 - x^2 - y^2)$

$$\sigma_x = (1, 0, -2x)$$
$$\sigma_y = (0, 1, -2y)$$

SO WE GET: $\sigma_x \times \sigma_y = (2x, 2y, 1)$, $dS = \|\sigma_x \times \sigma_y\| \, dx\,dy = (4x^2+4y^2+1)^{\frac{1}{2}} dx\,dy$

$$\iint_R x(4x^2+4y^2+1) \, dx\,dy$$

WHERE R = CIRCLE RADIUS 2 CENTERED AT $(0,0)$ SINCE THIS IS WHERE OUR SURFACE INTERSECTS THE XY-PLANE.

$\downarrow$ CONVERT TO POLAR

$$= \int_0^{2\pi} \int_0^2 r\cos\theta(4r^2+1) \, r \, dr \, d\theta$$

$$= \iint 4r^4\cos\theta + r^2\cos\theta \, dr \, d\theta$$

$$= \int \frac{4}{5} r^5 \cos\theta + \frac{1}{3} r^3 \cos\theta \Big|_0^2 \, d\theta$$

$$= \int \left(\frac{128}{5} + \frac{8}{3}\right) \cos\theta \, d\theta$$

$$= \left(\frac{128}{5} + \frac{8}{3}\right) \sin\theta \Big|_0^{2\pi} = 0$$

NOTICE THAT IN THE CASE OUR SURFACE IS OF THE FORM $z = f(x,y)$ THAT WE MAY CHOOSE:

$$\sigma(x,y) = (x, y, f(x,y))$$

SO $\sigma_x = (1, 0, f_x)$

$\sigma_y = (0, 1, f_y)$

$\sigma_x \times \sigma_y = (-f_x, -f_y, 1)$ SO $dS = \|\sigma_x \times \sigma_y\| \, dx\,dy$

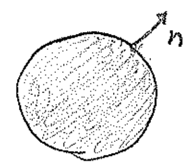
AND $$\iint g \, dS = \iint g \cdot \sqrt{(f_x)^2 + (f_y)^2 + 1} \; dx\,dy$$

SIMILAR FORMULAS HOLD FOR IF OUR SURFACE IS GIVEN AS $y = f(x,z)$, $x = f(y,z)$.

FLUX INTEGRALS

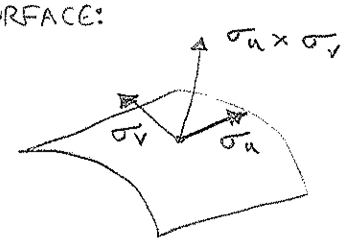
FOR SOME VECTOR FIELD $F = (P, Q, R)$ AND SOME CLOSED SURFACE (MEANING IT IS THE BOUNDARY OF SOME 3-DIMENSIONAL THING - FOR EXAMPLE THE UNIT SPHERE $x^2 + y^2 + z^2 = 1$) THEN THE FLUX (OR FLOW) OF F IN OR OUT OF OUR SURFACE IS:

$$\iint F \cdot n \, dS$$



WHERE n IS AN OUTWARD UNIT NORMAL VECTOR.

IF WE PARAMETRIZE OUR SURFACE $\sigma(u,v)$, THEN THE VECTORS σ_u AND σ_v ARE TANGENT VECTORS TO OUR SURFACE:



⊥ MEANS "IS PERPENDICULAR TO"

THUS $\sigma_u \times \sigma_v$ IS A NORMAL VECTOR (SINCE $\sigma_u \times \sigma_v \perp \sigma_u$, AND $\perp \sigma_v$)

THUS $n = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ OR $-\dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$, WHICHEVER IS THE OUTWARD NORMAL

$$\iint F \cdot n \, dS = \pm \iint F \cdot \frac{\sigma_u \times \sigma_v}{\cancel{\|\sigma_u \times \sigma_v\|}} \cancel{\|\sigma_u \times \sigma_v\|} \, du\,dv = \pm \iint F \cdot (\sigma_u \times \sigma_v) \, du\,dv$$

CHOOSE SIGN APPROPRIATELY

WE CAN STILL COMPUTE FLUX ALONG NONCLOSED SURFACES AS LONG AS WE SPECIFY WHAT NORMAL VECTOR TO USE.

EX: COMPUTE $\iint F \cdot n \, dS$ ALONG THE PORTION OF THE PLANE $z = 2 + x + 2y$ ABOVE THE SQUARE $0 \le x \le 1$, $0 \le y \le 1$, WITH $F = (x, y+z, z)$ AND n THE NORMAL W/ POSITIVE z-COORDINATE.

$$\sigma(x,y) = (x, y, 2+x+2y)$$

$$\sigma_x = (1, 0, 1)$$

$$\sigma_y = (0, 1, 2)$$

$\sigma_x \times \sigma_y = (-1, -2, 1)$ AND $n = \dfrac{(-1,2,1)}{\|(-1,2,1)\|}$ HAS + Z-COORD SO IT IS THE CORRECT n

$$\iint F \cdot n \, dS = \iint (x, y+z, z) \cdot \frac{\sigma_x \times \sigma_y}{\cancel{\|\sigma_x \times \sigma_y\|}} \cancel{\|\sigma_x \times \sigma_y\|} \, dx\,dy$$

$$= \iint (x, y+z, z) \cdot (-1, -2, 1) \, dx\,dy$$

$$= \iint -x - 2y - 2z + z \, dx\,dy$$

$$= \iint -x - 2y - z \, dx\,dy \longrightarrow$$ $z = 2 + x + 2y$ ALONG OUR SURFACE (WHICH IS WHERE WE EVALUATE F)

$$= \iint -x - 2y - (2 + x + 2y) \, dx\,dy$$

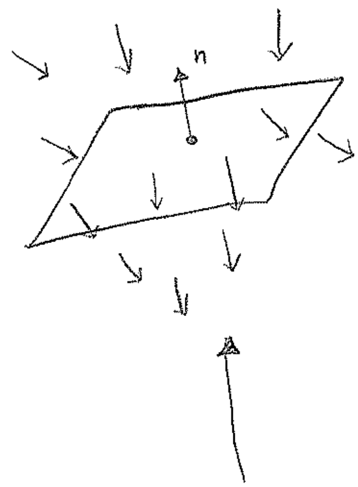
$$= \int_0^1 \int_0^1 -2x - 4y - 2 \, dx\,dy$$

$$= \int -x^2 - 4xy - 2x \Big|_0^1 \, dy$$

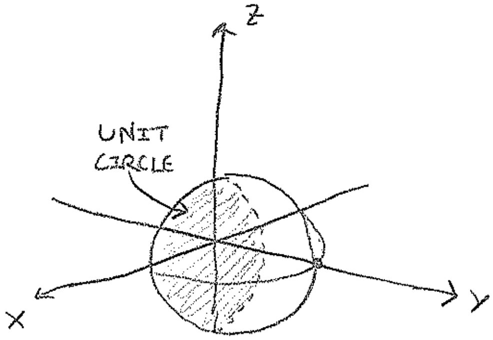
$$= \int -3 - 4y \, dy$$

$$= -3y - 2y^2 \Big|_0^1$$

$= -5$ SO MORE IS "FLOWING" IN A DIRECTION OPPOSING n.



EX: COMPUTE THE FLUX ALONG THE PART OF THE PARABOLOID $Y = 1 - X^2 - Z^2$ WITH $Y \geq 0$ AND NORMAL VECTOR WITH POSITIVE Y COMPONENT, $F = (X, Y, Z)$.



HERE $\sigma(x,z) = (x,\ 1 - x^2 - z^2,\ z)$ WILL BE OUR PARAMETRIZATION

$$\sigma_x = (1, -2x, 0)$$

$$\sigma_z = (0, -2z, 1)$$

$\sigma_x \times \sigma_z = (-2x, -1, -2z)$ HAS <u>NEGATIVE</u> Y-COMPONENT SO...

$n = -\dfrac{\sigma_x \times \sigma_z}{\|\sigma_x \times \sigma_z\|}$ NOTICE THE $-$ SIGN SO THAT OUR UNIT NORMAL IS CORRECT!

$$\iint F \cdot n \, dS = \iint (x,y,z) \cdot \left(-\frac{\sigma_x \times \sigma_z}{\cancel{\|\sigma_x \times \sigma_z\|}}\right) \cancel{\|\sigma_x \times \sigma_z\|} \, dx\, dz$$

$$= \iint (x,y,z) \cdot (2x, 1, 2z) \, dx \, dz$$

$$= \iint 2x^2 + y + 2z^2 \, dx \, dz \longrightarrow \text{ON OUR SURFACE } y = 1 - x^2 - z^2$$

$$= \iint x^2 + z^2 + 1 \, dx dz \longrightarrow \text{WE INTEGRATE X \& Z OVER THE UNIT CIRCLE (SEE PICTURE) SO USE POLAR COORDS}$$

$$= \int_0^{2\pi}\int_0^1 (r^2+1) r \, dr \, d\theta$$

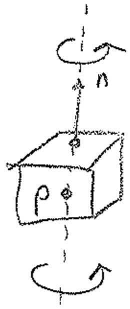
$$= \int \frac{1}{4}r^4 + \frac{1}{2}r^2 \Big|_0^1 \, d\theta$$

$$= \frac{3}{4}\int_0^{2\pi} d\theta$$

$$= \frac{3\pi}{2}$$

STOKES' THM

RECALL THAT $\text{curl}\,F \cdot n$ MEASURED HOW MUCH F ROTATED SOME INFINITELY SMALL CUBE AT A POINT ABOUT THE AXIS n:

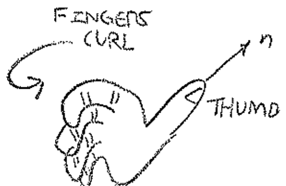


$(\text{curl}\,F\cdot n)(p) > 0$

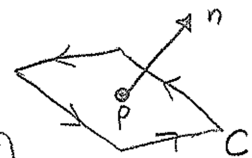
$(\text{curl}\,F\cdot n)(p)$ IS REALLY A <u>CIRCULATION PER UNIT AREA</u> SO WE MULTIPLY BY AN AREA ELEMENT & INTEGRATE IT IN STOKES' THM

ANOTHER WAY TO THINK OF IT IS TO THINK OF A SMALL FLAT SQUARE (CALL ITS BOUNDARY C) FLAKE CENTERED AT p WITH NORMAL VECTOR n. THEN $(\text{curl}\,F\cdot n)(p)\,dS = \oint_C F\cdot dr$ (CIRCULATION AROUND S)

dS: AREA OF SQUARE



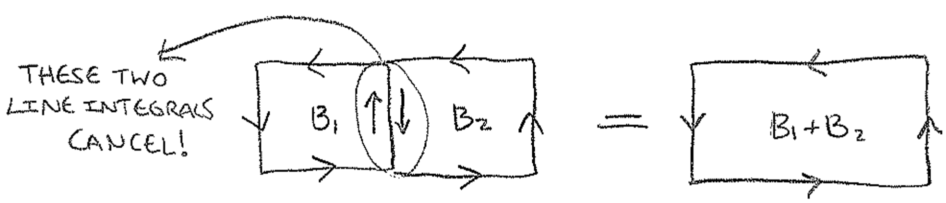
(I HOPE YOUR RIGHT HAND DOES NOT LOOK LIKE THIS)



WE USE THE <u>RIGHT HAND RULE</u> TO DECIDE IN WHAT DIRECTION TO GO ON C.

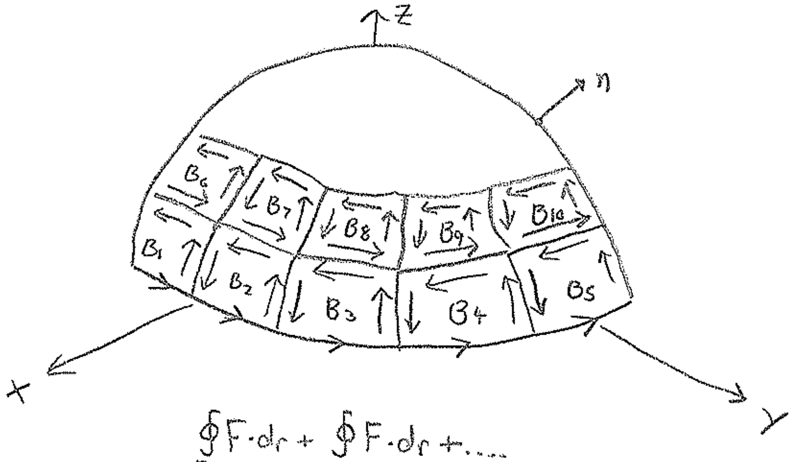
(OF COURSE, THIS SQUARE SHOULD BE INFINITELY SMALL)

SO WHAT WE WILL DO IS SPLIT UP OUR SURFACE INTO BLOCKS JUST AS WE DID TO COMPUTE SURFACE INTEGRALS AND ADD UP ALL OF THESE CIRCULATION INTEGRALS. THEN, ADJACENT SIDES OF OUR BLOCKS HAVE CANCELLING LINE INTEGRALS:



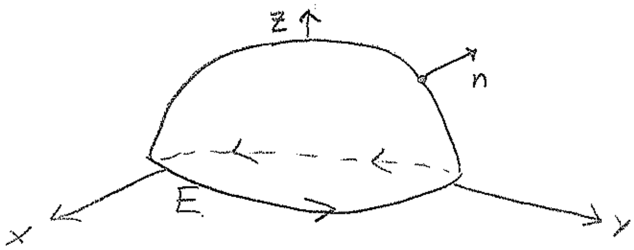
$$\oint_{B_1} F\cdot dr + \oint_{B_2} F\cdot dr \qquad \oint_{B_1+B_2} F\cdot dr$$

SO WHEN WE HAVE SOMETHING LIKE A HEMISPHERE & CHOP IT UP INTO BLOCKS, AND SUM UP THE CIRCULATIONS AROUND EACH, WE WOULD EXPECT THE ONLY LINE INTEGRALS LEFT AFTER CANCELLING WILL BE THOSE ON THE "EQUATOR":



$$\oint_{B_1} F\cdot dr + \oint_{B_2} F\cdot dr + \dots$$

$$\|$$



$$\oint_E F\cdot dr \qquad E = \text{"EQUATOR"}$$

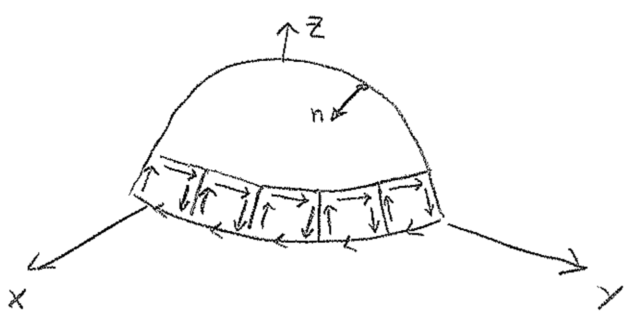
THIS IS THE IDEA OF STOKES' THM. WE JUST INSTEAD THINK OF THE CIRCULATION AROUND A SMALL BLOCK AS $(\text{curl} F \cdot n)\,dS$ AS DISCUSSED BEFORE.

<u>THM</u> (STOKES') IF S IS SOME SURFACE WHOSE BOUNDARY IS SOME CURVE C THEN:

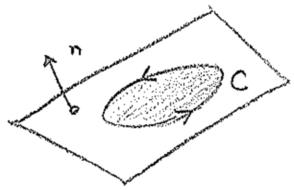
$$\iint_S (\text{curl} F \cdot n)\,dS = \oint_C F\cdot dr$$

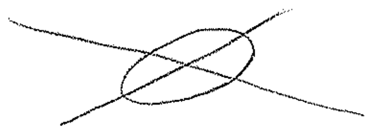
WHERE n IS THE UNIT NORMAL CONSISTENT WITH THE DIRECTION OF OUR PATH C.

IN OUR ABOVE EXAMPLE OF THE HEMISPHERE WE USED THE OUTWARD POINTING n AND FOUND OUR CURVE C TO BE TRAVERSED IN $\circlearrowleft$ DIRECTION IN THE XY-PLANE.
HAD WE USED THE INWARD POINTING NORMAL OUR C WOULD BE $\circlearrowright$:



EX: COMPUTE THE CIRCULATION INTEGRAL AROUND THE PIECE OF THE PLANE $z = 8 + x + 2y$ ABOVE THE ELLIPSE $\frac{x^2}{4} + \frac{y^2}{9} = 1$ IN THE XY-PLANE (TRAVERSED IN THE COUNTERCLOCKWISE DIRECTION), IF $F = (x-2y,\ z+3x,\ y-z)$.





WE USE STOKES':

$$\oint_C F \cdot dr = \iint \text{curl}\, F \cdot n \, dS$$

$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-2y & z+3x & y-z \end{vmatrix} = (0, 0, 5)$$

OUR SURFACE IS GIVEN BY $\sigma(x,y) = (x, y, 8+x+2y)$

$$\sigma_x = (1, 0, 1)$$

$$\sigma_y = (0, 1, 2)$$

$\sigma_x \times \sigma_y = (-1, -2, 1)$ HAS + z COMPONENT, WILL GIVE CORRECT n

SO $n = \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|}$, $dS = \|\sigma_x \times \sigma_y\|\, dx\, dy$

(THESE ALWAYS CANCEL IN THESE TYPES OF INTEGRALS)

$$\iint \text{curl}\, F \cdot n \, dS = \iint (0,0,5) \cdot \frac{\sigma_x \times \sigma_y}{\cancel{\|\sigma_x \times \sigma_y\|}} \cancel{\|\sigma_x \times \sigma_y\|}\, dx\, dy$$

$$= \iint (0,0,5) \cdot (-1,-2,1)\, dx\, dy$$

$$= 5 \iint dx\, dy \longrightarrow 5\,(\text{AREA OF ELLIPSE})$$ SEE PG 163

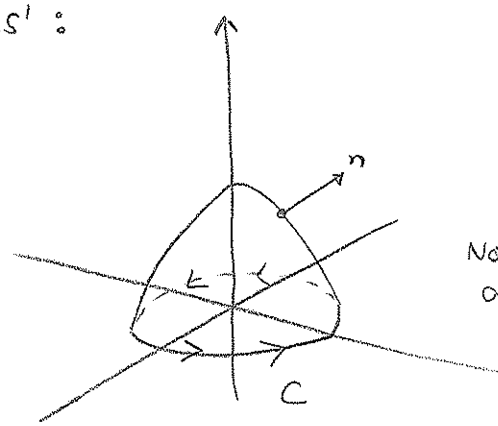
$$= 5(\pi(2)(3))$$

$$= 30\pi$$

DOING THE LINE INTEGRAL DIRECTLY W/ $r(t) = (2\cos t,\ 3\sin t,\ 8 + 2\cos t + 6\sin t)$ WILL GET US THE RIGHT ANSWER BUT WILL BE A LITTLE MESSIER.

EX: COMPUTE $\iint \text{curl}\,F \cdot n\, dS$ ON THE PARABOLOID $z = 4 - x^2 - y^2$, $z \geq 0$ WITH $F = (ze^{x+y} + x,\ y + z^2\cos x,\ z + xy)$ AND NORMAL W/ POSITIVE Z-COMPONENT.

FIRST NOTE THAT DOING THIS DIRECTLY WILL BE UGLY (CURL F WE BE INTENSE) SO INSTEAD APPLY STOKES':



NOTE THAT THE DIRECTIONS OF n AND C ARE CONSISTENT

$$\iint \text{curl}\,F \cdot n\, dS = \oint_C F \cdot dr$$

$r(t) = (2\cos t, 2\sin t, 0)$ IS C ($z=0$ IN $z = 4 - x^2 - y^2$)

$$= \int_0^{2\pi} (\cancel{ze^{x+y}} + x,\ y + \cancel{z^2\cos x},\ \cancel{z} + xy) \cdot (2\cos t, 2\sin t, 0)\, dt$$

$z = 0$ CANCELS OUT STUFF!

$$= \int_0^{2\pi} (2\cos t,\ 2\sin t,\ 4\cos t\sin t) \cdot \underbrace{(-2\sin t,\ 2\cos t,\ 0)}_{r'(t)}\, dt$$

$$= \int_0^{2\pi} -4\cos t\sin t + 4\cos t\sin t\, dt$$

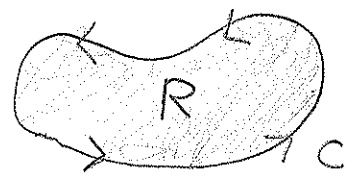
$$= 0$$

AGAIN NOTE HOW MUCH SIMPLER ONE WAY IS THAN THE OTHER! THIS IS THE BIGGEST CHALLENGE IN THESE PROBLEMS – NOT ONLY DO YOU NEED TO KNOW HOW TO DO THEM BUT ALSO LEARN TO PICK OUT THE EASIEST WAY TO DO IT.

BACK TO GREEN'S THM

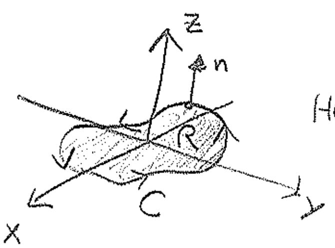
NOW WE SHOW HOW GREEN'S THM IS JUST STOKES'.

SUPPOSE WE HAVE SOME CURVE C AROUND A REGION R IN $\mathbb{R}^2$ AND $F=(P,Q)$ IS OUR VECTOR FIELD.



GREEN'S THM SAYS: $\displaystyle\oint_C F\cdot dr = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$

NOW THINK OF THIS CURVE IN $\mathbb{R}^3$ AND $F=(P,Q,0)$ TO BE THE EXTENSION OF F TO A VECTOR FIELD IN $\mathbb{R}^3$. THEN WE CAN APPLY STOKES THM TO THE SURFACE R:



HERE $n=(0,0,1)$ OVER ALL OF R

STOKES' SAYS:
$$\oint_C F\cdot dr = \iint_R \text{CURL} F\cdot n\, dS$$

$\longrightarrow dS = dx\,dy$ ON THE XY-PLANE

$\sigma(x,y)=(x,y,0) \Longrightarrow \|\sigma_x\times\sigma_y\| = 1$

$$= \iint_R \text{CURL} F\cdot(0,0,1)\,dx\,dy$$

$$= \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{z \text{ COMPONENT OF CURL } F} dx\,dy$$

WHICH IS JUST GREEN'S THM.

THUS GREEN'S THM IS JUST STOKES' THM WHEN OUR SURFACE IS FLAT & STUCK ON THE XY-PLANE.