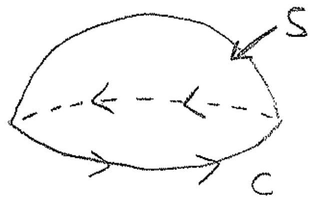
LAST TIME: WE TALKED ABOUT SURFACE INTEGRALS AND STOKES' THM WHICH PUT THE CIRCULATION OF SOME VECTOR FIELD F ALONG A CURVE IN TERMS OF A SURFACE INTEGRAL OF A SURFACE WHOSE BOUNDARY IS THAT CURVE:



$$\oint_C F \cdot dr = \iint_S \text{curl}\, F \cdot n \, dS$$

SO STOKES' THM RELATES CERTAIN INTEGRALS ON A TWO-DIMENSIONAL OBJECT TO INTEGRALS ALONG ITS BOUNDARY. WE WILL COVER THE DIVERGENCE THEOREM TODAY WHICH RELATES CERTAIN INTEGRALS ALONG THREE-DIMENSIONAL OBJECTS TO INTEGRALS ALONG THEIR BOUNDARIES. FIRST, A REVIEW OF:

TRIPLE INTEGRALS & CYLINDRICAL/SPHERICAL COORDINATES

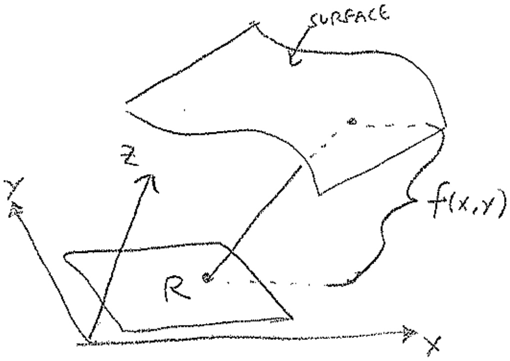
WE CAN USE TRIPLE INTEGRALS TO COMPUTE VOLUMES AND INTEGRATE FUNCTIONS OVER 3-DIMENSIONAL DOMAINS:

$$\text{Volume} = \iiint dx\,dy\,dz \qquad \text{MASS} = \iiint f \, dx\,dy\,dz$$

f = DENSITY FUNCTION

WE CALL $dV = dx\,dy\,dz$ THE "VOLUME ELEMENT"

WHEN WE USED DOUBLE INTEGRALS TO COMPUTE THE VOLUME UNDER A SURFACE $z = f(x,y)$ WE WERE REALLY DOING A TRIPLE INTEGRAL:

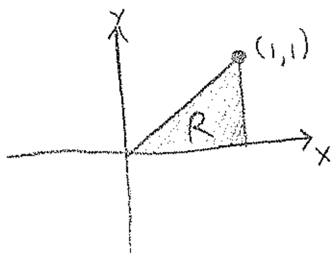


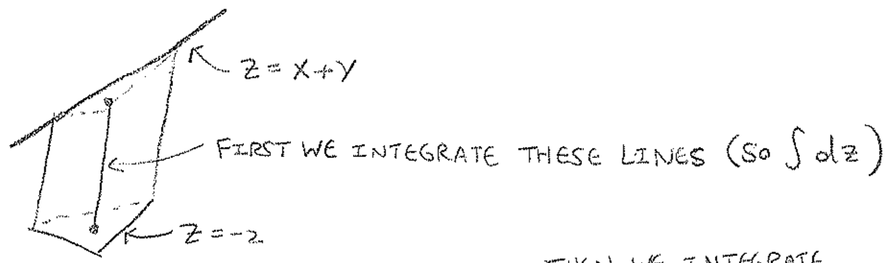
$$V = \iint_R f(x,y)\, dx\,dy$$

$$= \iint_R \int_0^f dz \, dx\,dy$$

When figuring out the limits of the integration, picture our integration as occurring one variable at a time starting at the innermost integral, thus forming 1-dimensional strips which we integrate along in the second variable to get 2-dimensional slices and so on:

EX: Compute the volume of the space between $y=0$, $x=1$, $y=x$, $z=-2$, and $z=x+y$





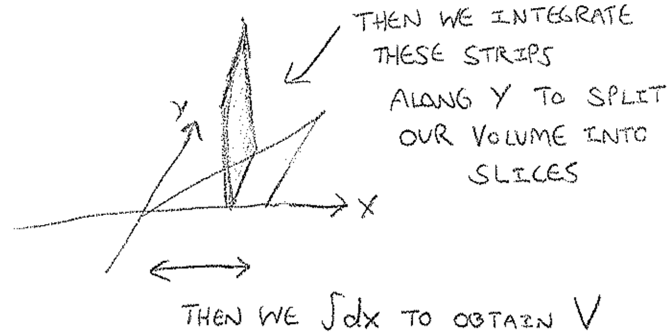
$$V = \int_0^1 \int_0^x \int_{-2}^{x+y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^x x+y+2 \, dy\, dx$$

$$= \int_0^1 y(x+2) + \tfrac{1}{2}y^2 \Big|_0^x dx$$

$$= \int_0^1 \tfrac{3}{2}x^2 + 2x \, dx$$

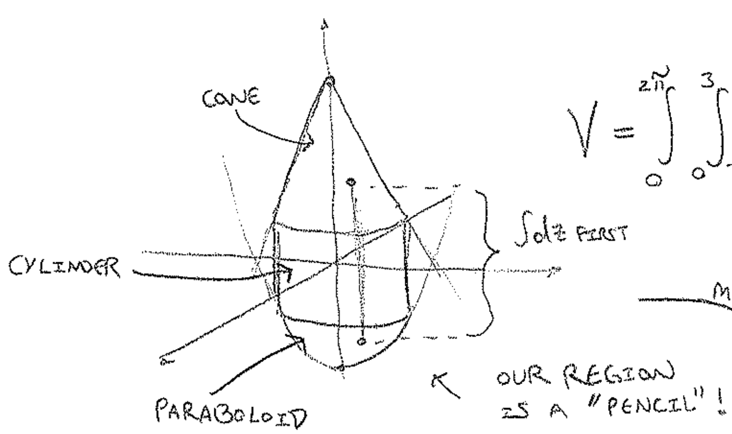
$$= \tfrac{1}{2}x^3 + x^2 \Big|_0^1 = \frac{3}{2}$$



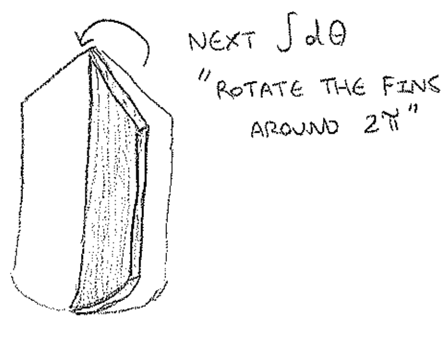
Cylindrical coordinates in $\mathbb{R}^3$: $(\underbrace{r, \theta}_{\text{usual polar coords in } \mathbb{R}^3}, z)$

EX: Compute the volume between the surfaces $x^2+y^2=9$, $z=-\sqrt{x^2+y^2}+4$, $z=x^2+y^2-16$
$\uparrow$ cylinder $\quad \uparrow$ cone $\quad \uparrow$ paraboloid

$$V = \int_0^{2\pi} \int_0^3 \int_{-\sqrt{x^2+y^2}+16}^{\sqrt{x^2+y^2}+16} dz\, r\, dr\, d\theta$$



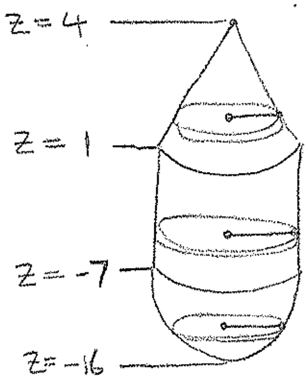
Then $\int dr$ making these 2-D "fins"



NOTICE THAT SINCE $dx\,dy = r\,dr\,d\theta$ IN POLAR, IN CYLINDRICAL COORDINATES WE HAVE $dV = r\,dz\,dr\,d\theta$.

NOW WHAT IF WE WANT THE SAME VOLUME BUT CHANGE OUR ORDER OF INTEGRATION?

SAY WE WANT $V = \iiint r\,dr\,d\theta\,dz$



WE SPLIT THE DOMAIN INTO 3 PIECES:

$r = 0$ TO $r = $ CONE CONE EQ: $z = -\sqrt{x^2+y^2}+4 = -r+4$ SO $r = 4 - z$ IS CONE

$r = 0$ TO $r = 3$

$r = 0$ TO $r = $ PARABOLOID PARAB. EQ.: $z = x^2+y^2-16 = r^2-16$ SO $r = \sqrt{z+16}$ IS PARABOLOID

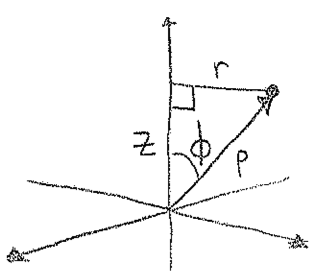
THEN $$V = \int_{-16}^{-7}\int_0^{2\pi}\int_0^{\sqrt{z+16}} r\,dr\,d\theta\,dz + \int_{-7}^{1}\int_0^{2\pi}\int_0^{3} r\,dr\,d\theta\,dz + \int_{1}^{4}\int_0^{2\pi}\int_0^{4-z} r\,dr\,d\theta\,dz$$

NOTICE HOW COMPLICATED THINGS GET IF WE CHOOSE A POOR ORDER OF INTEGRATION!

SPHERICAL COORDINATES IN $\mathbb{R}^3$: (ρ, θ, ϕ)

ρ = DISTANCE FROM $(0,0,0)$, θ = SAME, ϕ = ANGLE W/ POSITIVE Z-AXIS

SO WE HAVE THE FOLLOWING PICTURE:



WHICH IMPLIES: $\cos\phi = \dfrac{z}{\rho}$ SO $z = \rho\cos\phi$

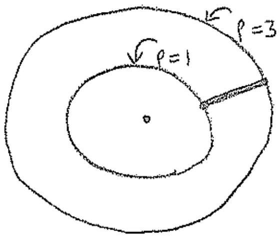
$\sin\phi = \dfrac{r}{\rho}$ SO $r = \rho\sin\phi$

$z^2 + r^2 = \rho^2$ SO $x^2+y^2+z^2 = \rho^2$

IN SPHERICAL COORDINATES, OUR VOLUME ELEMENT BECOMES:

$$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

EX: COMPUTE THE VOLUME BETWEEN THE SPHERES $x^2+y^2+z^2=1$ AND $x^2+y^2+z^2=9$.



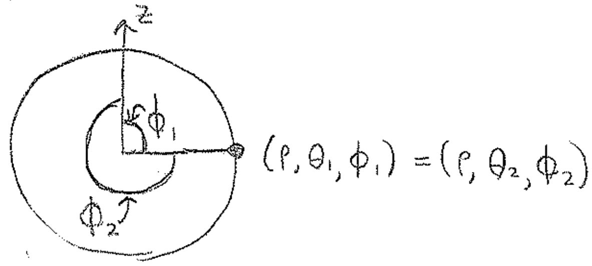
$$V = \int_0^{2\pi}\int_0^{\pi}\int_1^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{3}\rho^3\Big|_1^3 \iint \sin\phi \, d\phi \, d\theta$$

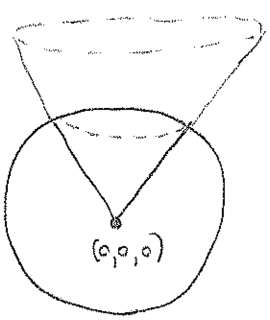
$$= 9(-\cos\phi)\Big|_0^{\pi} \int d\theta$$

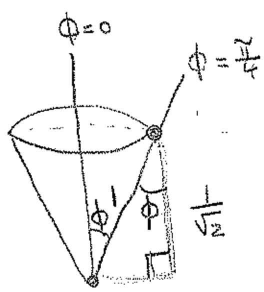
$$= 18(2\pi) = 36\pi$$

NOTE: ϕ ONLY RANGES FROM <u>0 TO π</u> SINCE OTHERWISE WE WOULD DOUBLE COVER OUR SPHERE:



EX: COMPUTE THE VOLUME BETWEEN THE SPHERE $x^2+y^2+z^2=1$ AND CONE $z^2=x^2+y^2$





HERE WE FIND THAT THESE SURFACES INTERSECT WHEN $2z^2=1$ (PLUG IN $z^2=x^2+y^2$ INTO EQ. 1)

$$z = \pm\frac{1}{\sqrt{2}}$$

THEN WE HAVE THE Δ TO THE LEFT, SO $\cos\phi = \frac{1}{\sqrt{2}}$ AT OUR POINTS OF INTERSECTION, WHICH IMPLIES <u>$\phi = \frac{\pi}{4}$</u>

THUS $$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{3}\rho^3\Big|_0^1 \left(-\cos\phi\right)\Big|_0^{\frac{\pi}{4}} \theta\Big|_0^{2\pi}$$

$$= \frac{2\pi}{3}\left(-\frac{1}{\sqrt{2}}+1\right)$$

NOW THAT WE'VE REVIEWED TRIPLE INTEGRALS, IT IS TIME FOR:

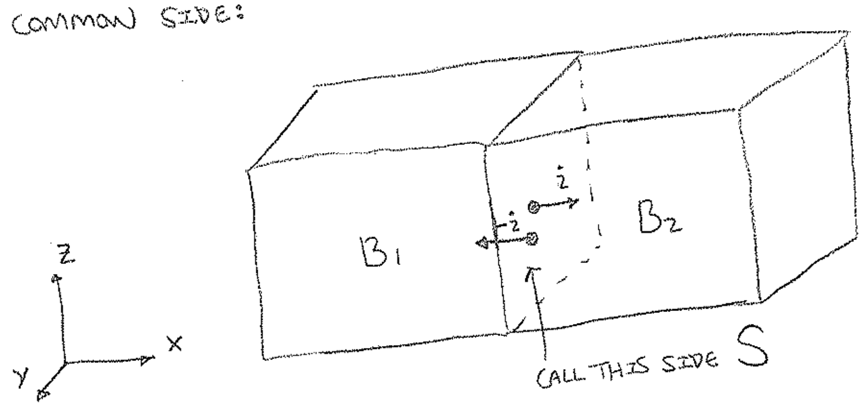
DIVERGENCE THEOREM

RECALL THAT FOR A VECTOR FIELD $F = (P, Q, R)$ WAS DEFINED AS:

$$\operatorname{DIV} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

AND IT MEASURED FLOW PER UNIT VOLUME OF AN INFINITELY SMALL BOX. (SEE 140)

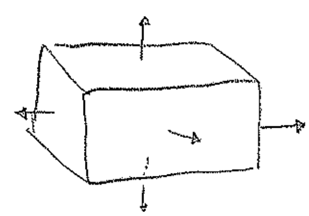
NOTICE THAT IF WE COMPUTE THE FLUX INTEGRALS $\iint F \cdot n \, dS$ AROUND ADJACENT BOXES, THAT WE HAVE CANCELLATION OF THE SURFACE INTEGRALS ALONG THE COMMON SIDE:



SO $\hat{i}$ IS THE OUTWARD NORMAL FOR B_1 ALONG S, $-\hat{i}$ THE OUTWARD NORMAL FOR B_2 ALONG S.

$$\left.\begin{array}{l} \displaystyle\iint_{B_1} F \cdot n \, dS = \iint_{S} F \cdot \hat{i} \, dS + \iint_{\substack{\text{OTHER} \\ \text{SIDES} \\ \text{OF } B_1}} F \cdot n \, dS \\ \displaystyle\iint_{B_2} F \cdot n \, dS = \iint_{S} F \cdot (-\hat{i}) \, dS + \iint_{\substack{\text{OTHER} \\ \text{SIDES} \\ \text{OF } B_2}} F \cdot n \, dS \end{array}\right\} \text{ADD (\& CANCEL FIRST TERMS)}$$

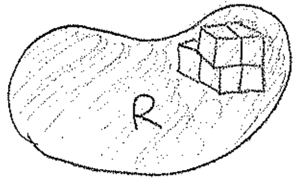
$$\iint_{B_1} F \cdot n \, dS + \iint_{B_2} F \cdot n \, dS = \iint_{B_1 + B_2} F \cdot n \, dS$$



SO WHAT WE DO IS TAKE SOME 3-D DOMAIN AND SPLIT IT UP INTO BLOCKS.

FOR BLOCKS SMALL ENOUGH, $\underset{\substack{\uparrow \\ \text{VOLUME OF BLOCK}}}{\operatorname{DIV} F(P) \, dV} = \underbrace{\iint_B F \cdot n \, dS}_{\text{FLUX IN/OUT OF } B}$ FOR A BLOCK B CENTERED AT P.

CALL OUR DOMAIN R.



SO ADDING THESE UP OVER ALL BLOCKS:

$$\underbrace{\text{DIV}\, F(P_1)\, dV_1 + \text{DIV}\, F(P_2)\, dV_2 + \dots}_{\iiint_R \text{DIV}\, F\, dV} = \underbrace{\iint_{B_1} F\cdot n\, dS + \iint_{B_2} F\cdot n\, dS + \dots}_{\iint F\cdot n\, dS}$$

DIVERGENCE THEOREM

$$\iiint_R \text{DIV}\, F\, dV = \iint_{\text{BOUNDARY OF } R} F\cdot n\, dS$$

INTUITIVELY IT STATES THAT THE FLOW IN/OUT OF R ONLY DEPENDS ON WHAT FLOWS IN/OUT OF THE BOUNDARY OF R

NOTATION: ∂R MEANS "BOUNDARY OF THE REGION R"

LET'S NOW USE THIS THM:

EX: COMPUTE $\iint_{\partial R} F\cdot n\, dS$ WHERE R IS THE CUBE GIVEN BY $0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2$ AND $F = (xy,\ z^2+y^2,\ x^2)$

WITHOUT THE DIVERGENCE THM WE WOULD NEED TO DO <u>SIX</u> SURFACE INTEGRALS TO COMPUTE THIS! INSTEAD WE HAVE:

$$\iint_{\partial R} F\cdot n\, dS = \iiint_R \text{DIV}\, F\, dV$$

$$= \iiint y + 2y + 0\, dV$$

$$= 3\iiint y\, dV$$ WE USE $dx\,dy\,dz$ SINCE OUR DOMAIN IS A CUBE

$$= 3\int_0^2\int_0^2\int_0^2 y\, dx\, dy\, dz$$

$$= 6\iint y\, dy\, dz$$

$$= 3\int y^2\Big|_0^2\, dz$$

$$= 12(2) = 24$$

NOT ONLY IS THE DIVERGENCE THEOREM USEFUL FOR REGIONS WITH SEVERAL BOUNDARY SURFACES, BUT COMPLICATED FIELDS MAY HAVE EASILLY-INTEGRATED DIVERGENCES:

EX: COMPUTE $\iint F \cdot n \, dS$ ALONG THE EXTERIOR OF THE UNIT SPHERE WITH $F = (x^3 + e^{yz},\ y^3 + \cos x,\ z^3 - \sin y)$

(LET R = UNIT SPHERE)

$$\iint_{\partial R} F \cdot n \, dS = \iiint_R \operatorname{DIV} F \, dV$$

$$= \iiint_R 3x^2 + 3y^2 + 3z^2 \, dV \longrightarrow \text{CONVERT TO SPHERICAL}$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \left(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \right)$$

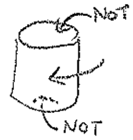
$$= \frac{3}{5} \rho^5 \Big|_0^1 \iint \sin\phi \, d\phi \, d\theta$$

$$= \frac{3}{5} \left(-\cos\phi \right) \Big|_0^{\pi} \int d\theta$$

$$= \frac{3}{5} (1 + 1) \, 2\pi$$

$$= \frac{12\pi}{5}$$

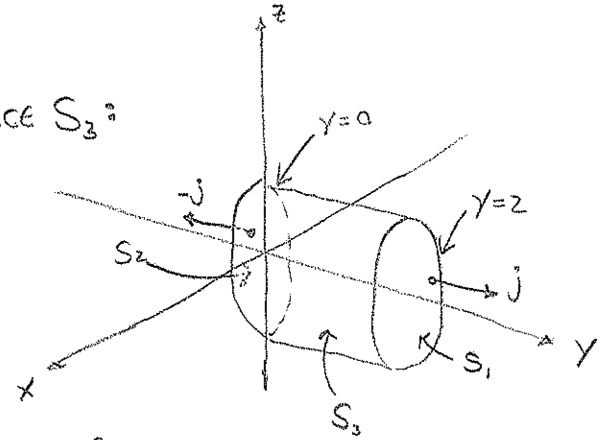
LIKE WHEN APPLYING GREEN'S THM TO COMPUTE LINE INTEGRALS WHEN OUR LINE INTEGRAL WASN'T A LOOP, IN WHICH CASE WE ADD CURVES TO PRODUCE A LOOP TO APPLY GREEN'S THM, WE CAN TAKE SURFACES THAT ARE NOT OF THE FORM ∂R AND ADD MORE SO THAT WE CAN APPLY THE DIVERGENCE THEOREM.

EX: COMPUTE $\iint F\cdot n\, dS$ ALONG THE SIDES 

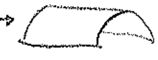
OF THE CYLINDER CUT OUT BY

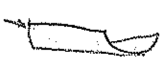
$x^2+z^2=4$, $y=0$, $y=2$. $\quad F=(y\cos z+3x,\ 1,\ ye^x+yz)$

CALL OUR "SIDES" SURFACE S_3:



FIRST NOTE THAT TO COMPUTE $\iint_{S_3} F\cdot n\, dS$ WE WOULD NEED TO USE <u>TWO SURFACE INTEGRALS</u> IF X AND Y WERE OUR PARAMETERS:

$\sigma_1=(x,y,\sqrt{4-x^2}) \rightarrow$ 

$\sigma_2=(x,y,-\sqrt{4-x^2}) \rightarrow$ 

ALSO, JUST IMAGINE $F\cdot(\sigma_x\times\sigma_y)$, AND THEN IMAGINE INTEGRATING IT.

INSTEAD WE ADD THE SURFACES S_1 & S_2 AS PICTURED AND APPLY THE DIVERGENCE THEOREM:

$$\overset{\text{I}}{\iint_{S_1} F\cdot n\, dS}+\overset{\text{II}}{\iint_{S_2} F\cdot n\, dS}+\iint_{S_3} F\cdot n\, dS=\overset{\text{III}}{\iiint_R \operatorname{div} F\, dV}$$

WHERE R = REGION INSIDE OUR CYLINDER. NOW WE COMPUTE THE NUMERALED TERMS:

I THIS IS THE CIRCULAR PART OF THE PLANE $y=2$. IT HAS OUTWARD NORMAL $n=j$ AT EVERY POINT. THUS:

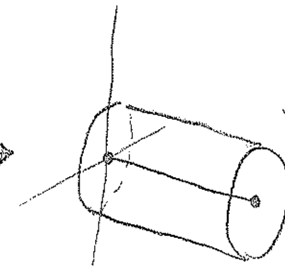
$$\iint_{S_1} F\cdot n\, dS=\iint_{S_1} F\cdot j\, dS=\iint_{S_1} dS=4\pi \quad \text{(AREA OF CIRCLE)}$$

II HERE $n=-j$ SO $\iint_{S_2} F\cdot n\, dS=\iint_{S_2} F\cdot(-j)\, dS=-\iint_{S_2} dS=-4\pi$

THUS I + II $=0$ AND OUR ANSWER WILL BE III

III $\displaystyle\iiint_R \text{DIV}\, F \, dV = \iiint 3 + 0 + y \, dV$

$$= 3 \underbrace{\iiint dV}_{\text{VOLUME OF}} + \iiint y \, dV$$



$\int dy$ FIRST THEN OVER ALL X,Z IN THE CIRCLE RADIUS 2

$$= 3(4\pi)(2) + \iint \int_0^2 y \, dy \, dx \, dz$$

$$= 24\pi + \iint \tfrac{1}{2} y^2 \Big|_0^2 \, dx \, dz$$

$$= 24\pi + 2 \underbrace{\iint dx \, dz}_{\text{AREA OF CIRCLE RADIUS 2}}$$

$$= 24\pi + 2(4\pi)$$

$$= 32\pi$$

SO OUR ANSWER IS

<u>FACT</u> $\text{DIV}(\text{CURL}\, F) = 0$ FOR <u>ANY</u> $F = (P, Q, R)$

$\text{DIV}(\text{CURL}\, F) = \text{DIV}(R_y - Q_z, -R_x + P_z, Q_x - P_y)$ → HERE I USE THE SHORTHAND $P_y = \frac{\partial P}{\partial y}$

$$= \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(-R_x + P_z) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= \cancel{R_{xy}} - \cancel{Q_{xz}} - \cancel{R_{xy}} + \cancel{P_{yz}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

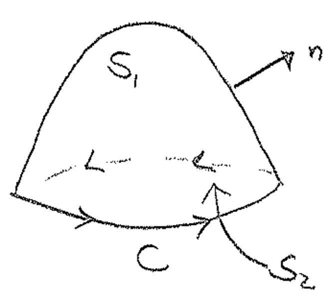
$$= 0$$

LET'S USE THIS IN AN EXAMPLE:

EX: SUPPOSE $\text{CURL}\, F = (-1 - e^z, 1, x^2)$. THEN COMPUTE $\displaystyle\iint_{S_1} \text{CURL}\, F \cdot n \, dS$

WHERE S_1 IS THE SURFACE CONSISTING OF THE PARABOLOID $z = 1 - x^2 - y^2$, $z \geq 0$,

AND n IS THE NORMAL WITH POSITIVE Z-COORDINATE:



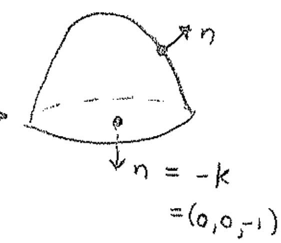
DIRECT COMPUTATION WILL AGAIN BE ROUGH. WE COULD TRY AND APPLY STOKES' THM:

$$\iint_{S_1} \text{curl}\, F \cdot n \, dS = \oint_C F \cdot dr$$

BUT WE DON'T KNOW F! INSTEAD WE CONSTRUCT THE SURFACE S_2 (BOTTOM PART) AND APPLY THE DIVERGENCE THEOREM TO THE FIELD curl F:

$$\iint_{S_1} \text{curl}\, F \cdot n \, dS + \iint_{S_2} \text{curl}\, F \cdot n \, dS = \iiint \underbrace{\text{div}(\text{curl}\, F)}_{=0} \, dV = 0$$

ON S_2, $n = (0,0,-1)$



$$\text{THUS} \quad \iint_{S_1} \text{curl}\, F \cdot n \, dS = -\iint_{S_2} \text{curl}\, F \cdot n \, dS$$

$$= -\iint \text{curl}\, F \cdot (0,0,-1) \, dS$$

$$= \iint x^2 \, dx \, dy \longrightarrow \text{INTEGRATE OVER UNIT CIRCLE}$$

$$= \int_0^{2\pi}\int_0^1 r^3 \cos^2\theta \, dr \, d\theta$$

$$= \frac{1}{4} r^4 \Big|_0^1 \int \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

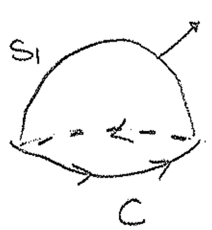
$$= \frac{1}{4}\left(\frac{1}{2}\right)\left(\theta + \cancel{\frac{1}{2}\sin 2\theta}\right)\Big|_0^{2\pi}$$

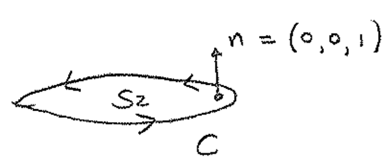
$$= \frac{\pi}{4}$$

NOTE THAT WE ACTUALLY WOULD HAVE GOTTEN THIS USING STOKES' THM (TWICE!)

$$\iint_{S_1} \text{curl}\, F \cdot n \, dS = \oint_C F \cdot dr = \iint_{S_2} \text{curl}\, F \cdot (0,0,1) \, dS \quad \text{AS ABOVE}$$

STOKES' STOKES'





THIS SHOWS HOW DEEPLY RELATED ALL OF THESE THM'S ARE (RECALL GREEN'S IS JUST A SPECIAL CASE OF STOKES')

EX: SUPPOSE R IS SOME CONNECTED 3-DIMENSIONAL REGION IN $\mathbb{R}^3$.

SUPPOSE THAT THE FLUX INTEGRAL $\iint_{\partial R} F \cdot n \, dS = 8$, WHERE WE HAVE

$F = (e^z \cos y + 3x, \; y - \sin x, \; \cos(xy))$.

FIND THE VOLUME OF R.

HMMM TRICKY!

APPLY THE DIVERGENCE THM:

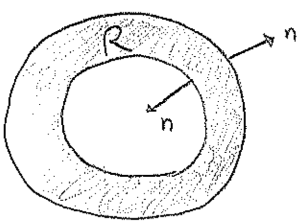
$$8 = \iint_{\partial R} F \cdot n \, dS = \iiint_R \operatorname{div} F \, dV = \iiint_R 4 \, dV = 4 \, (\text{VOLUME OF } R)$$

THUS (VOLUME OF R) = 2.

EX: IF R IS THE REGION BETWEEN THE SPHERES OF RADIUS 2 AND 3, CENTERED AT $(0,0,0)$ COMPUTE

$\iint_{\partial R} F \cdot n \, dS$ FOR $F = (xy^2 + \cos z, \; yz^2, \; x^2 z + y)$

IS THE SUM OF <u>TWO</u> SURFACE INTEGRALS (ONE ON INTERIOR, ONE ON EXTERIOR)



$$\iint_{\partial R} F \cdot n \, dS = \iiint_R \operatorname{div} F \, dV$$

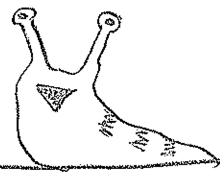
$$= \iiint_R \underbrace{y^2 + z^2 + x^2}_{\rho^2} \, dV \longrightarrow \text{CONVERT TO POLAR}$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{5}\rho^5 \Big|_2^3 \; (-\cos\phi) \Big|_0^{\pi} \; \theta \Big|_0^{2\pi}$$

$$= \frac{1}{5}(3^5 - 32)(2)(2\pi)$$

$$= \frac{4\pi}{5}(3^5 - 32)$$



THE END