LAST TIME:

WE TALKED ABOUT
- LINEAR COMBS AND SPAN (BUG!)
- LINEAR INDEPENDENCE (2 EQUIVALENT DEFINITIONS)
- BASIS - IN A VECTOR SPACE OF DIMENSION n, A BASIS IS A SET OF n LINEARLY INDEPENDENT VECTORS.

MORE ABOUT MATRICES

DEF: IF A IS AN $m \times n$ MATRIX, ITS TRANSPOSE A^T IS AN $n \times m$ MATRIX WHOSE ROWS ARE THE COLUMNS OF A (IN THE SAME ORDER).

EX: $\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

PROPERTIES

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ (NOTICE WE SWAPPED THE ORDER)
- $(cA)^T = cA^T \quad c \in \mathbb{R}$

IF THE PRODUCT AB MAKES SENSE, A IS $m \times n$, B IS $n \times p$ FOR SOME m, n, p

SO: A^T IS $n \times m$, B^T IS $p \times n$

AND THE ONLY WAY IT MAKES SENSE TO MULTIPLY THESE IS $\underset{p \times n}{B^T} \underset{n \times m}{A^T}$

DEF:
WE SAY THAT A MATRIX A IS <u>SYMMETRIC</u> IF $A = A^T$.

IF A IS $m \times n$ AND SYMMETRIC,

$$\underset{m \times n}{A} = \underset{n \times m}{A^T} \quad \text{SO} \quad m = n$$

DEF: A MATRIX WITH THE SAME # OF ROWS AS COLUMNS IS CALLED A <u>SQUARE MATRIX</u>

EX: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}$

THESE ARE SYMMETRIC MATRICES

WE CAN THINK OF THE MATRICES AS BEING SYMMETRIC ALONG THE DIAGONAL:

$$\begin{pmatrix} 1 & -3 & 0 \\ -3 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

HENCE THE NAME

<u>MAGIC TRICK:</u>

PICK ANY 2×2 MATRIX, (SAY $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ IS PICKED)

THEN I WILL FIND A 2×2 MATRIX A s.t. $A \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ IS <u>SYMMETRIC</u>.

LET $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. THEN:

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1+1 & 2-3 \\ 2-3 & 4+9 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 13 \end{pmatrix} \text{ IS SYMMETRIC.}$$

HOW?? NOTICE THE MATRIX I PICKED WAS JUST THE TRANSPOSE OF $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

IN GENERAL, THE MATRIX AA^T IS SYMMETRIC FOR <u>ANY</u> A.

WHY? LET'S RECALL THE DEFINITION.

<u>A MATRIX B IS SYMMETRIC IFF $B^T = B$</u>

LET'S CHECK:

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$(A^T)^T = A$

BY DISTRIBUTIVE PROPERTY (SWAP ORDER!)

SO AA^T IS SYMMETRIC! NOW GO AMAZE YOUR FRIENDS!

DEF: THE <u>MAIN DIAGONAL</u> OF A MATRIX A IS

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ THIS PART.

WE SAY A IS <u>DIAGONAL</u> IF IT IS ZERO EVERYWHERE BUT ON THE MAIN DIAGONAL: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ ARE ALL DIAGONAL

A MATRIX A IS <u>UPPER TRIANGULAR</u> IF IT IS ZERO BELOW THE MAIN DIAGONAL: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ ARE ALL U.T.

A MATRIX A IS <u>LOWER TRIANGULAR</u> IF IT IS ZERO ABOVE THE MAIN DIAGONAL: $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ ARE L.T.

THE DIAGONAL MATRICES $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, ETC

ARE CALLED <u>IDENTITY MATRICES</u> USUALLY DENOTED I

THE REASON FOR THIS IS THAT IF A IS $m \times n$,

$$\underset{m\times m}{I}\;\underset{m\times n}{A} = A \qquad \text{AND} \qquad A = \underset{m\times n}{A}\;\underset{n\times n}{I}$$

EX: SUPPOSE $A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{pmatrix}$

COMPUTE: ① C^TB ② BA^T ③ $(C^T + A)B$

① $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1(-1) + 0(2) + 0(1) \\ 0(-1) + 2(2) + (-1)(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$2 \times 3 \quad 3 \times 1$

ANSWER IS 2×1

② $BA^T = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix}^T = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ -1 & 2 \end{pmatrix}$

$3 \times 1 \quad 3 \times 2$

CAN'T MULTIPLY!

③ $(C^T + A)B = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \right] \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3 & -1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$2 \times 3 \quad 3 \times 1$

$= \begin{pmatrix} 2(-1) + 3(2) + (-1)(1) \\ 0(-1) + 3(2) + 1(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

MATRICES AS FUNCTIONS

RECALL WE CAN CONSIDER AN $m \times n$ MATRIX A AS A FUNCTION (PG 13) TAKING VECTORS IN $\mathbb{R}^n$ TO VECTORS IN $\mathbb{R}^m$, WRITTEN:

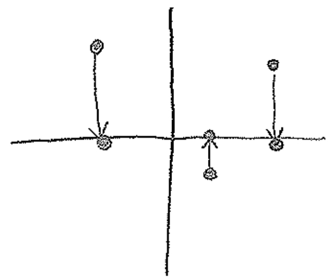
$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

LET'S DO SOME 2×2 EXAMPLES

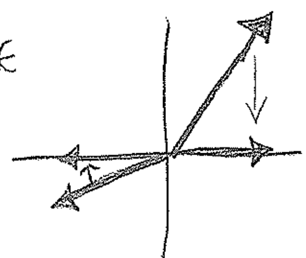
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{THEN} \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

SO OUR FUNCTION IS $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ 0 \end{pmatrix}$ (PROJECTION ONTO X-AXIS)

WE CAN THINK OF $\begin{pmatrix} x \\ y \end{pmatrix}$ AS A POINT IN $\mathbb{R}^2$, IN WHICH CASE OUR PICTURE IS LIKE THIS:



OR AS A VECTOR IN WHICH CASE OUR PICTURE IS LIKE:



$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{THEN} \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

SO OUR FUNCTION IS $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

THIS IS A 90° ROTATION SO $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

THE ANGLE BETWEEN $\begin{pmatrix} x \\ y \end{pmatrix}$ AND $\begin{pmatrix} -y \\ x \end{pmatrix}$ IS 90°

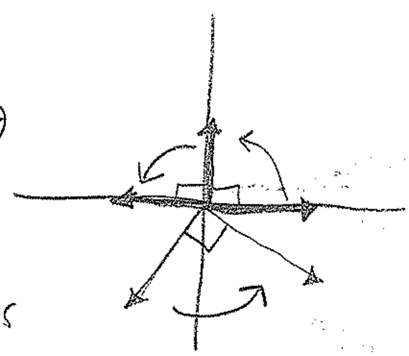
SINCE THE DOT PRODUCT:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix} = -xy + xy = 0 = \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| \left\|\begin{pmatrix} -y \\ x \end{pmatrix}\right\| \cos\theta$$

$$0 = \cos\theta$$

$$\frac{3\pi}{2} \text{ OR } \frac{\pi}{2} = \theta$$

SEE PG 4 FOR DOT PRODUCTS



SYSTEMS OF EQUATIONS

SOLVING A SYSTEM OF LINEAR EQUATIONS:

$$3x + y - z = 1$$

$$x + y + 2z = 0$$

$$-x - y + z = 0$$

FOR $x, y,$ AND z CAN BE THOUGHT OF AS FINDING A VECTOR $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s.t.

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

WE WILL WRITE SYSTEMS LIKE THIS OFTEN. THE GENERAL FORM IS GIVEN A MATRIX A AND VECTOR b, CAN WE FIND A VECTOR x SUCH THAT $Ax = b$?

WE CAN SOLVE A SYSTEM OF EQUATIONS BY USING THE FOLLOWING ELEMENTARY OPERATIONS.

1. MULTIPLY AN EQUATION BY SOME $c \in \mathbb{R}$ $(c \neq 0)$ $\quad E_1 \longrightarrow 2E_1$ (DOUBLE EQ 1)
2. SWAP THE ORDER OF TWO OF THE EQUATIONS $\quad E_1 \longleftrightarrow E_2$ (SWAP EQ 1 & 2)
3. ADD ANY MULTIPLE OF ONE EQUATION TO ANOTHER. $\quad E_1 \longrightarrow E_1 + 3E_2$ (ADD $3E_2$ TO E_1)

THE KEY IS THAT ALL OF THESE OPERATIONS PRESERVE THE SOLUTIONS

EX: SOLVE $3x+3y=9$

$x-y=7$

$$\downarrow \; E_1 \leftrightarrow E_2 \text{ (SWAP)}$$

$x-y=7$

$3x+3y=9$

$$\downarrow \; E_2 \longrightarrow E_2 - 3E_1$$

$x-y=7$

$0+6y=-12$

$$\downarrow \; E_2 \longrightarrow \tfrac{1}{6}E_2$$

$x-y=7$

$0+y=-2$

$$\downarrow \; E_1 \longrightarrow E_1 + E_2$$

$x+0=7$

$0+y=-2$ SO $x=7 \quad y=-2$

FOR EASE OF NOTATION, WE TYPICALLY OMIT THE VARIABLES AND $+/-$'S AND INSTEAD WRITE OUR SYSTEM OF EQUATIONS AS:

$$\left(\begin{array}{cc|c} 3 & 3 & 9 \\ 1 & -1 & 7 \end{array}\right) \text{ MEANS } \left(\begin{array}{c|c} 3x+3y & 9 \\ x-y & 7 \end{array}\right)$$

THIS IS CALLED THE <u>AUGMENTED MATRIX</u> CORRESPONDING TO THE SYSTEM OF EQUATIONS. OUR ELEMENTARY OPERATIONS THEN BECOME OPERATIONS ON THE ROWS OF THIS MATRIX:

TO SOLVE A SYSTEM IN AN AUGMENTED MATRIX, WE CAN DO <u>ELEMENTARY ROW OPERATIONS</u> TO FIND THE SOLUTION

1. MULTIPLY A ROW BY SOME $c \in \mathbb{R}$ $(c \neq 0)$
2. SWAP TWO ROWS
3. ADD MULTIPLES OF ROWS TO OTHER ROWS

EX: SOLVE $\begin{aligned} 2x + 4y &= 0 \\ x - y &= 3 \end{aligned}$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & -1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 6 & -6 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{6}R_2} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

SAYS $\rightarrow$ $\begin{aligned} x + 0y &= 2 \\ 0x + y &= -1 \end{aligned}$ SO $\boxed{\begin{aligned} x &= 2 \\ y &= -1 \end{aligned}}$

NOTICE WE'RE DOING THE SAME THING BUT JUST WRITING LESS BY USING AUGMENTED MATRICES.

IN BOTH OF OUR EXAMPLES, THE SOLUTION WAS ONE POINT.

IN GENERAL, THERE MAY BE INFINITE SOLUTIONS OR NO SOLUTIONS.

A SYSTEM $Ax = b$ IS CONSISTENT IF IT HAS A SOLUTION

IT IS INCONSISTENT IF IT DOES NOT.

EX: SOLVE

$$\begin{aligned} x - 2y + z &= 1 \\ y + 3z &= 2 \\ 2x - 4y + 2z &= 8 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & -4 & 2 & 8 \end{array}\right)$$

$$\Big\downarrow \; R_3 \to R_3 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

LOOK AT THE LAST ROW. THIS CORRESPONDS TO THE EQUATION $0x + 0y + 0z = 6$ OR SIMPLY

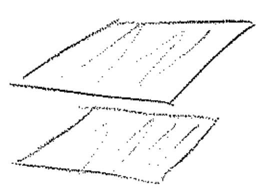
$$0 = 6$$

THIS CANNOT HAPPEN! THUS THERE ARE NO SOLUTIONS AND THE SYSTEM IS INCONSISTENT

RMK: EVERY INCONSISTENT SYSTEM'S AUGMENTED MATRIX CAN BE TRANSFORMED SO THAT WE GET A ROW READING $(0 \; 0 \; 0 \dots 0 \mid c)$ FOR SOME NONZERO NUMBER c.

GEOMETRICALLY THE FIRST AND THIRD EQ'S WERE FOR PARALLEL PLANES:

$x - 2y + z = 1$ AND $x - 2y + z = 4$ (DIVIDED BY 2)



SO THEY DON'T INTERSECT AND NO SOLUTION EXISTS

THE USUAL WAY TO SOLVE A SYSTEM $\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right)$ IS TO ROW REDUCE IT INTO A NICE FORM CALLED ROW-ECHELON FORM:

DEF: A MATRIX A IS IN ROW-ECHELON FORM IF

1. THE FIRST NONZERO ENTRY IN EACH ROW IS A 1
2. IN CONSECUTIVE ROWS, THE 1 IN THE LOWER ROW IS FURTHER RIGHT
3. ALL ZERO ROWS ARE AT THE BOTTOM OF THE MATRIX

EX: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ ARE IN ROW-ECHELON FORM

$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ <u>IS NOT</u> (NEED TO SWAP $R_2 \leftrightarrow R_3$)

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ <u>IS NOT</u> (AGAIN SWAP $R_2 \leftrightarrow R_3$)

ONCE WE HAVE AN AUGMENTED MATRIX IN ROW-ECHELON FORM, IT BECOMES STRAIGHTFORWARD TO SOLVE.

THE <u>GENERAL STRATEGY</u> TO PUT A MATRIX IN R-E FORM IS

WORK 1 COLUMN AT A TIME. FIRST GET A 1 IN THE TOP OF THE FIRST COLUMN, THEN GET ZEROS ALL BELOW IT. THEN KEEP MOVING RIGHT WORKING COLUMN BY COLUMN. MORE EXAMPLES WILL DEMONSTRATE THIS.

EX: SOLVE $2x+7y-9z=-4$

$x+2y=1$

$-y+3z=2$

1ST COLUMN DONE!

$$\left(\begin{array}{ccc|c} 2 & 7 & -9 & -4 \\ 1 & 2 & 0 & 1 \\ 0 & -1 & 3 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 7 & -9 & -4 \\ 0 & -1 & 3 & 2 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 3 & -9 & -6 \\ 0 & -1 & 3 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 3 & -9 & -6 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to -R_2}$$

2ND COLUMN DONE!

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ IS NOW IN ROW-ECHELON FORM

NOW START AT THE BOTTOM EQ AND GO UP:

E_2: $y-3z=-2$ SO

$y=-2+3z$

E_1:

AND $x+2y=1$

$x=1-2y=1-2(-2+3z)=-3z+5$

SO OUR SOLUTIONS ARE $\begin{pmatrix} -3z+5 \\ -2+3z \\ z \end{pmatrix}$ WHERE z CAN BE ANYTHING.

WE CAN REWRITE THIS AS $\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ THIS IS A LINE IN 3-SPACE

WE SAY THAT THE SOLUTION SPACE HAS 1 PARAMETER IN THIS CASE. NOTE THAT WE HAVE INFINITELY MANY SOLUTIONS. WHAT WE FOUND IS THAT THESE 3 PLANES INTERSECT ALONG A LINE.

EX:
$$\left(\begin{array}{cccc|c} -1 & 2 & 2 & 0 & 4 \\ 1 & -2 & -2 & 0 & -4 \\ 0 & 1 & 3 & 1 & 1 \\ 1 & -1 & 1 & 1 & -3 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \left(\begin{array}{cccc|c} -1 & 2 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 \end{array}\right) \xrightarrow{R_4 \to R_4 - R_3}$$

$$\left(\begin{array}{cccc|c} -1 & 2 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} -1 & 2 & 2 & 0 & 4 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

THIS IS A SYSTEM OF 4 EQS AND 4 VARIABLES SO LETS CALL THEM X, Y, Z, W

THEN E_2: $Y + 3Z + W = 1$

$$Y = 1 - W - 3Z$$

E_1: $-X + 2Y + 2Z = 4$

$$2Y + 2Z - 4 = X \quad \text{PLUG IN FOR } Y$$

$$2(1 - W - 3Z) + 2Z - 4 = X$$

$$-2 - 4Z - 2W = X$$

OUR SOLUTIONS ARE

$$\begin{pmatrix} -2-4Z-2W \\ 1-3Z-W \\ Z \\ W \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Z \begin{pmatrix} -4 \\ -3 \\ 1 \\ 0 \end{pmatrix} + W \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

HERE WE HAVE 2 PARAMETERS

HOW DO WE KNOW HOW MANY PARAMETERS WE WILL HAVE? IN THIS CASE, WE HAD 4 VARIABLES AND EACH OF THE 2 EQUATIONS OF THE MATRIX IN ROW-ECHELON FORM LET US PUT ONE VARIABLE IT TERMS OF THE OTHERS. SO A GOOD GUESS WOULD BE:

PARAMETERS = # VARIABLES − # ROWS (NON ZERO!) OF THE AUGMENTED MATRIX IN ROW-ECHELON FORM

THIS IS TRUE. IT IS VERY GOOD TO KNOW. ESPECIALLY FOR A TEST AND/OR QUIZ.

WE NOW MAKE UP A WORD FOR THIS LAST TERM IN THE ABOVE FORMULA:

DEF: THE RANK OF A MATRIX A IS THE NUMBER OF NONZERO ROWS IN ITS ROW-ECHELON FORM

SO THE ABOVE EQ IS:

$$\#\text{ PARAMETERS OF AUGMENTED SYSTEM } (A \mid b) = \#\text{ OF VARIABLES } \left(\text{SAME AS } \#\text{ COLUMNS OF } A\right) - \text{RANK}(A)$$

EX: FIND RANK OF $\begin{pmatrix} 1 & -2 & 0 \\ 7 & -9 & -5 \\ 1 & 1 & -3 \end{pmatrix}$

DO ROW OPS: $\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 7R_1} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 5 & -5 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

THIS HAS 2 NONZERO ROWS SO THE RANK IS 2

WE USUALLY DENOTE RANK(A) BY RK(A).

THE RANK HAS MORE GEOMETRIC INTERPRETATIONS WHICH WE WILL DISCUSS NOW. FIRST SOME DEFINITIONS.

DEF: THE <u>ROW SPACE</u> OF A MATRIX A WITH ROW VECTORS $r_1, \dots, r_m$ IS THE SPAN $\{r_1, \dots, r_n\}$, AND IS DENOTED R_A.

EX: IF $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$, $R_A = \text{SPAN}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\} = \text{SPAN}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

<u>NOTICE</u> THAT THE SPAN $\{V_1, \dots, V_n\}$ OF ANY VECTORS $V_1 \dots V_n$ IS A VECTOR SPACE! RECALL THE 3 THINGS WE NEED:

1. IF W_1 AND W_2 ARE IN SPAN $\{V_1, \dots, V_n\}$, SO IS $W_1 + W_2$
2. IF W_1 IS IN SPAN $\{V_1, \dots, V_n\}$, SO IS cW_1 FOR ANY $c \in \mathbb{R}$
3. THE ZERO VECTOR IS IN SPAN $\{V_1, \dots, V_n\}$

THESE ARE ALL TRUE:

1. $W_1 = a_1V_1 + a_2V_2 + \dots + a_nV_n$
   $W_2 = b_1V_1 + b_2V_2 + \dots + b_nV_n$ a's AND b's $\in \mathbb{R}$

   THEN $\underline{W_1 + W_2} = (a_1+b_1)V_1 + (a_2+b_2)V_2 + \dots + (a_n+b_n)V_n \in \underline{\text{SPAN}\{V_1, \dots, V_n\}}$

2. $\underline{cW_1} = c(a_1V_1 + \dots + a_nV_n) = (ca_1)V_1 + \dots + (ca_n)V_n \in \underline{\text{SPAN}\{V_1, \dots, V_n\}}$

3. $0V_1 + 0V_2 + \dots + 0V_n = 0 \in \text{SPAN}\{V_1, \dots, V_n\}$

SO IT MAKES SENSE TO TALK ABOUT THE DIMENSION OF SPAN $\{V_1, \dots, V_n\}$. (WE DEFINED DIMENSION OF A VECTOR SPACE ON PAGE 9).

DEF: THE COLUMN SPACE OF A MATRIX A WITH COLUMNS $C_1, \dots, C_n$ IS THE SPAN $\{C_1, \dots, C_n\}$ AND IS DENOTED C_A.

BIG THM:

$$RK(A) = \dim(R_A) = \dim(C_A)$$

THE KEY FACT IS THE COLUMN SPACE C_A HAS A VERY NICE GEOMETRIC MEANING.

CONSIDER $A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$

THEN $$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ 2x + 4y \end{pmatrix} = \underbrace{x\begin{pmatrix} 3 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 4 \end{pmatrix}}_{\text{LINEAR COMB. OF COLUMNS OF } A} \in C_A$$

☆ THIS SHOWS THAT ANY VECTOR OF THE FORM $A\begin{pmatrix} x \\ y \end{pmatrix}$ IS IN C_A, AND CONVERSELY ANY VECTOR IN C_A IS OF THE FORM $A\begin{pmatrix} x \\ y \end{pmatrix}$.

DEF: THE IMAGE OF A (SAY A IS $m \times n$) IS THE SET OF ALL VECTORS Av FOR ALL $v \in \mathbb{R}^n$:

$$\operatorname{Im}(A) = \{ Av \mid v \in \mathbb{R}^n \}$$

IF WE THINK OF A AS A FUNCTION (SEE PG 13) THEN THE IMAGE OF A IS THE "STUFF A HITS" IN $\mathbb{R}^m$.

THE CONTENT OF ☆ ABOVE IS THAT:

$$\operatorname{Im} A = C_A$$

SO THE BIG THM SAY $RK(A) = \dim(C_A) = \dim(\operatorname{Im} A)$

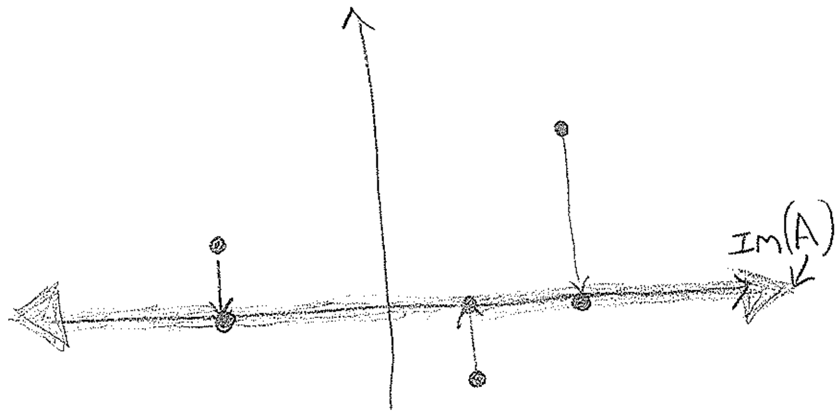
KEY EQUALITY

EX: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ IS 2×2 AND THUS DEFINES A FUNCTION:

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

GEOMETRICALLY, WE'RE JUST PROJECTING DOWN TO THE X-AXIS:



$$\text{IM}(A) = \left\{ A\begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; \text{''} \right\}$$

$$= \underline{\text{X-AXIS}}$$

THUS IM(A) IS <u>1-DIMENSIONAL</u> (IT IS A LINE)

SO $\text{DIM}(\text{IM}(A)) = 1$

A IS IN ROW-ECHELON FORM ALREADY, SO $\text{RK}(A) = \#\text{NONZERO ROWS}$
$= 1$

THUS WE HAVE $\text{RK}(A) = \text{DIM}(\text{IM}(A))$ AS SAID BEFORE

THIS IS HOW PEOPLE THINK OF THE RANK OF A GEOMETRICALLY –

IF A IS RANK 2, IM A IS <u>A PLANE</u>

IF A IS RANK 1, IM A IS <u>A LINE</u> ETC.