LAST TIME:

MAINLY TALKED ABOUT SOLVING SYSTEMS OF EQUATIONS.

LET'S RECALL THE METHOD: SUPPOSE WE WANT TO SOLVE $Ax=b$ (SYSTEM)

FIRST WE WRITE THE SYSTEM IN AN AUGMENTED MATRIX $(A|b)$

AND THEN ROW-REDUCE IT TO ROW-ECHELON FORM.

CASE 1: WE GET SOMETHING LIKE:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \text{OR} \quad \left(\begin{array}{cccc|c} 1 & 9 & -1 & -3 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

I.E. THE LAST EQ SAYS $0=1$ (BAD!).

IN THIS CASE THERE ARE <u>NO SOLUTIONS</u> AND WE SAY THE SYSTEM IS <u>INCONSISTENT</u>.

CASE 2: OTHERWISE, OUR SYSTEM WILL HAVE SOLUTIONS, (<u>CONSISTENT</u>) IT MAY HAVE JUST ONE OR INFINITE. THE # OF PARAMETERS TELLS US THE DIMENSION OF SOLUTIONS WE HAVE. RECALL:

$$\#\text{ PARAMETERS} = \#\text{ VARIABLES} - \underset{\uparrow}{\mathrm{RK}(A)}$$

(# OF NONZERO ROWS IN R.E.F.)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

#PAR = 3 − 3 = <u>0 PARAMS</u> (UNIQUE SOLUTION)

#PAR = 3 − 2 = <u>1 PARAM</u> (SOLUTION IS A LINE)

ETC.

NOW I WANT TO TALK A LITTLE MORE ABOUT RANK AND L.I.

RECALL I STATED THAT:

$$\mathrm{RANK}(A) = \mathrm{DIM}(\underset{\substack{\uparrow \\ \text{ROW} \\ \text{SPACE}}}{R_A}) = \mathrm{DIM}(\underset{\substack{\uparrow \\ \text{COLUMN} \\ \text{SPACE}}}{C_A}) \longrightarrow$$

NOTE THIS IMPLIES $\mathrm{RK}(A) = \mathrm{RK}(A^T)$ SINCE $R_A = C_{A^T}$

WE WILL USE THIS TO GET YET ANOTHER (BUT USEFUL) WAY TO THINK OF THE RANK.

EX: ARE THE VECTORS $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}$, AND $\begin{pmatrix}3\\5\\1\end{pmatrix}$ L.I.? (COMMON QUESTION)

RECALL OUR DEFINITION:

V_1, V_2, V_3 ARE L.I. IF $C_1V_1 + C_2V_2 + C_3V_3 = 0$ IMPLIES $C_1 = C_2 = C_3 = 0$

IN OTHER WORDS, THE ONLY WAY TO GET THE ZERO VECTOR AS A LINEAR COMBINATION OF V_1, V_2, AND V_3 IS TO NOT SUM ANY OF THEM!

(THINK OF i, j, AND k IN $\mathbb{R}^3$. THESE ARE L.I.)

SO SUPPOSE $c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}3\\5\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

WE WANT TO SHOW $c_1 = c_2 = c_3 = 0$

THIS REDUCES TO SOLVING:

$$c_1 + 3c_3 = 0$$
$$c_2 + 5c_3 = 0$$
$$c_1 + 2c_2 + c_3 = 0$$

USE AUGMENTED MATRICES...

(EQ'S OF PLANES THROUGH ORIGIN)

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2]{}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -12 & 0 \end{array}\right) \xrightarrow[R_3 \to -\frac{1}{12}R_3]{} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

R.E.F.
CONSISTENT

LOOK AT BOTTOM EQ FIRST:

E_3: $C_3 = 0$

E_2: $C_2 + 5C_3 = 0$ SO $C_2 = 0$

E_1: $C_1 + 3C_3 = 0$ SO $C_1 = 0$

ONLY SOLUTION IS $\boxed{C_1 = C_2 = C_3 = 0}$ SO THEY ARE L.I.

NOTE: OUR SYSTEM OF EQ'S WAS OF THE FORM $Ax = 0$

THIS SYSTEM IS CALLED HOMOGENEOUS. THE ZERO VECTOR $(x = 0)$ IS ALWAYS A SOLUTION IN THIS CASE. WHAT WE WERE DOING GEOMETRICALLY IS TAKING THE INTERSECTION OF 3 PLANES THAT ALL WENT THROUGH THE ORIGIN.

EX: ARE $\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$, AND $\begin{pmatrix}-2\\-3\\-1\end{pmatrix}$ L.I.?

SAME IDEA: SUPPOSE $c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}-2\\-3\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

WE WANT $c_1 = c_2 = c_3 = 0$

$$\left(\begin{array}{ccc|c}1&1&-2&0\\0&1&-3&0\\-1&0&-1&0\end{array}\right)\xrightarrow[R_3\to R_3+R_1]{}\left(\begin{array}{ccc|c}1&1&-2&0\\0&1&-3&0\\0&1&-3&0\end{array}\right)\xrightarrow[R_3\to R_3-R_2]{}$$

$$\left(\begin{array}{ccc|c}1&1&-2&0\\0&1&-3&0\\0&0&0&0\end{array}\right)$$

R.E.F.
CONSISTENT

#PARAMS $= 3 - 2 = \boxed{1}$ SO SOLUTION IS A LINE (NOT JUST $c_1=c_2=c_3=0$)

WE CAN SEE THIS THE USUAL WAY, LOOKING AT THE EQ'S:

$c_2 - 3c_3 = 0$

$\boxed{c_2 = 3c_3}$

$c_1 + c_2 - 2c_3 = 0$

$c_1 = -c_2 + 2c_3$

$c_1 = -(3c_3) + 2c_3$

$\boxed{c_1 = -c_3}$

SO OUR SOLUTIONS ARE

$\begin{pmatrix}-c_3\\3c_3\\c_3\end{pmatrix}$ FOR ANY c_3. SO LET'S CHOOSE $c_3 = 1$, SO $c_1 = -1$, $c_2 = 3$

AND CHECK: $(-1)\begin{pmatrix}1\\0\\-1\end{pmatrix} + (3)\begin{pmatrix}1\\1\\0\end{pmatrix} + (1)\begin{pmatrix}-2\\-3\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

THUS THE VECTORS ARE NOT L.I.

CLAIM: $\text{DIM}\left(\text{SPAN}\{v_1, .., v_n\}\right) =$ MAXIMUM # OF L.I. VECTORS IN THE SET $\{v_1, .., v_n\}$

RECALL $\text{SPAN}\{v_1, .., v_n\}$ IS A VECTOR SPACE AND THUS HAS A DIMENSION ASSOCIATED TO IT. WE <u>DEFINED</u> DIMENSION AS THE LARGEST # OF L.I. VECTORS ON PAGE ⑨, SO THIS IS ALMOST THE DEFINITION. IF WE ASSUME THIS, THEN:

$$\text{DIM}(R_A) = \text{DIM}\left(\text{SPAN}\{r_1, .., r_m\}\right)$$
$$\boxed{= \#\text{ OF L.I. ROWS OF } A}$$

WHERE A IS $m \times n$ MATRIX WITH ROW VECTORS $r_1, .., r_m$ AND COL. VECTORS $c_1, \dots, c_n$

AND:

$$\text{DIM}(C_A) = \text{DIM}\left(\text{SPAN}\{c_1, .., c_n\}\right)$$
$$\boxed{= \#\text{ OF L.I. COLUMNS OF } A}$$

AND BOTH OF THESE EQUAL $\text{RANK}(A)$ BY OUR <u>BIG THM</u> SO IN SUMMARY:

$$\boxed{\text{RK}(A) = \#\text{ OF L.I. ROWS OF } A = \#\text{ OF L.I. COLUMNS OF } A}$$

SO LET'S LOOK BACK AT THE PREVIOUS EXAMPLE.

EX: ARE $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, AND $\begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ L.I. ?

THEY ARE L.I. IFF THE COLUMNS OF THE MATRIX $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$ ARE L.I.

BUT # OF L.I. COLUMNS = $\text{RANK}(A)$. SO WE JUST ROW REDUCE TO FIND THE RANK: $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ HAS <u>RANK 2</u>. THUS THE VECTORS ARE NOT L.I.

BY THE SAME REASONING, THE VECTORS ARE L.I. IFF THE <u>ROWS</u> OF THE MATRIX $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -3 & -1 \end{pmatrix}$ ARE L.I.

BUT # OF L.I. ROWS = RANK(A). SO WE ROW REDUCE:

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ HAS <u>RANK 2</u>

SO OUR MATRIX HAS <u>2 L.I. ROWS</u> AND THUS THE VECTORS ARE NOT L.I.

SO NOTE WHAT WE SHOWED:

TO FIND IF A SET OF VECTORS $V_1, \ldots, V_n$ ARE L.I., WE CAN PUT THEM AS THE COLUMNS <u>OR</u> ROWS OF A MATRIX, AND THE RANK OF THAT MATRIX IS HOW MANY L.I. VECTORS WE HAVE IN $V_1, \ldots, V_n$.

IN PARTICULAR, IF IT IS RANK n THEY ARE L.I.

DETERMINANTS

FOR ANY SQUARE MATRIX A WE CAN TAKE A <u>DETERMINANT</u> OF IT, DENOTED $\det(A)$ OR SOMETIMES $|A|$, AND IT IS A REAL NUMBER.

<u>2×2 DETERMINANTS:</u>

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

LARGER DETERMINANTS ARE DEFINED IN TERMS OF DETERMINANTS OF DIMENSION ONE LESS. FIRST WE NEED A DEFINITION:

DEF: IF A IS $n \times n$, FOR ANY i AND j FROM 1 TO n WE HAVE THE <u>MINOR DETERMINANT</u> M_{ij}. IT IS $(n-1)\times(n-1)$ AND OBTAINED BY DELETING THE iTH ROW OF A AND jTH COLUMN OF A.

EX:

$$A = \begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix} \qquad M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

$$A = \begin{pmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & 8 & 9 \end{pmatrix} \qquad M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$$

DEF: THE <u>COFACTOR</u> DETERMINANT IS $C_{ij} = (-1)^{i+j} M_{ij}$

IF WE WRITE OUT THE MINORS OF LET'S SAY A 3×3 MATRIX IN A TABLE:

$$\begin{matrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{matrix}$$

THE COFACTORS ARE THE SAME BUT WE PUT NEGATIVE SIGNS IN A "CHECKERBOARD" PATTERN:

$$\begin{matrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{matrix} \quad = \quad \begin{matrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{matrix}$$

NOW FOR AN 3×3 MATRIX A, WE COMPUTE $\det(A)$ BY <u>EXPANSION BY COFACTORS</u>. FIRST LET'S EXPAND BY THE 1ST ROW:

NOTICE SAME INDICES

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boxed{a_{11} C_{11}} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

WE CAN EXPAND THIS ALONG <u>ANY ROW</u> AND WILL GET THE SAME ANSWER:

$$\text{DET}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

WE CAN ALSO EXPAND ALONG <u>ANY COLUMN</u>:

$$\text{DET}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

EX:

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 2C_{11} + 3C_{12} + \cancel{0C_{13}}$$

$$= 2M_{11} - 3M_{12}$$

$$= 2\begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 2(-4-6) - 3(-1)$$

$$= \boxed{-16}$$

NOTE HOW EXPANDING ALONG ROWS/COLS WITH MORE ZEROS = LESS WORK!

EX:

$$\begin{vmatrix} 0 & 1 & 0 & 3 \\ 1 & -1 & 4 & 2 \\ 0 & 3 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \cancel{0C_{31}} + 3C_{32} + \cancel{0C_{33}} - C_{34}$$

$$= -3M_{32} + M_{34}$$

$$= -3\begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 1 & -1 & 4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -3\left[0 - 0 + 3\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}\right] + \left[0 - \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 0\right]$$

$$= -3\left[3(-3)\right] + \left[-3\right]$$

$$= \boxed{30}$$

EX: $\begin{vmatrix} 2 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 0 + 0 = (2)(4)(6)$ = PRODUCT ALONG DIAGONAL

IN GENERAL:

IF A IS UPPER-TRIANGULAR (OR LOWER-TRIANGULAR) DET A = PRODUCT OF THE ELEMENTS ALONG THE MAIN DIAGONAL.

EX:
$$\begin{vmatrix} 1 & 7 & 13 & 101 & e^{\pi} & \sqrt{3} \\ 0 & 2 & -10 & 10 & \pi & -3 \\ 0 & 0 & 3 & 4 & -1 & -1 \\ 0 & 0 & 0 & 4 & 3 & -1 \\ 0 & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = \bigcirc\!\!\!\!\!\!\;\; 6!$$

SINCE WE CAN EXPAND ABOUT <u>ANY ROW OR COLUMN</u>, WE HAVE:

$$\text{DET}(A) = \text{DET}(A^T)$$

IF $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

EXPANDING ALONG ROW 1 OF A = EXPANDING ALONG COLUMN 1 OF A^T

PROPERTIES OF DETERMINANTS

MAIN ONE: ☆ $\text{DET}(AB) = \text{DET}(A)\,\text{DET}(B)$

THIS WILL IMPLY ALL OF THE OTHERS WILL WILL USE.

WE CAN DO ROW OPERATIONS ON DETERMINANTS, AND ALTHOUGH THE DETERMINANT DOES NOT ALWAY STAY THE SAME UNDER EACH OPERATION, IT CHANGES NICELY.

① SWAPPING ROWS INTRODUCES A NEGATIVE SIGN

$$\begin{vmatrix}1&2\\3&4\end{vmatrix} = -\begin{vmatrix}3&4\\1&2\end{vmatrix}$$

② MULTIPLYING ROWS BY SCALARS CAN BE THOUGHT OF AS "FACTORING A NUMBER OUT"

$$\begin{vmatrix}1&2\\3c&4c\end{vmatrix} = c\begin{vmatrix}1&2\\3&4\end{vmatrix}$$

FACTOR OUT C

③ ADDING A MULTIPLE OF ONE ROW TO ANOTHER <u>PRESERVES</u> THE DETERMINANT

$$\begin{vmatrix}1&2\\3&4\end{vmatrix} = \begin{vmatrix}1+3c&2+4c\\3&4\end{vmatrix}$$

NOW WHY ARE THESE THINGS TRUE? IT FOLLOWS FROM $\det(AB) \overset{\star}{=} \det(A)\det(B)$:
TO EVERY ROW OPERATION WE DO, THERE IS A MATRIX THAT WHEN YOU MULTIPLY ON THE LEFT BY IT, IT DOES THE <u>SAME THING</u>.

$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ SWAPS ROWS $\qquad \begin{pmatrix}1&0\\0&c\end{pmatrix}$ MULTIPLIES ROW 2 BY C

$\begin{pmatrix}1&c\\0&1\end{pmatrix}$ ADDS cR_2 TO R_1

NOW WE USE ☆

① $$\det\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\right) = \det\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3&4\\1&2\end{pmatrix}\right) \overset{\star}{=} \det\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)\det\left(\begin{pmatrix}3&4\\1&2\end{pmatrix}\right) = -1 \det\begin{pmatrix}3&4\\1&2\end{pmatrix}$$

② $$\det\left(\begin{pmatrix}1&2\\3c&4c\end{pmatrix}\right) = \det\left(\begin{pmatrix}1&0\\0&c\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\right) \overset{\star}{=} \det\left(\begin{pmatrix}1&0\\0&c\end{pmatrix}\right)\det\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\right) = c \det\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

③ $$\det\left(\begin{pmatrix}1+3c&2+4c\\3&4\end{pmatrix}\right) = \det\left(\begin{pmatrix}1&c\\0&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\right) \overset{\star}{=} \det\left(\begin{pmatrix}1&c\\0&1\end{pmatrix}\right)\det\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\right) = \det\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\right)$$

SO ROW REDUCTION IS JUST CLEVERLY DISGUISED MATRIX MULTIPLICATION!

WE USUALLY USE THE ROW-OPS TO GET OUR DETERMINANT INTO A NICE FORM (TRIANGULAR) SO THAT IT'S EASIER TO COMPUTE.

EX: $$\begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 5 & 7 & 2 \\ 3 & 7 & 2 & 1 \\ 2 & 1 & 0 & -1 \end{vmatrix} \xrightarrow{\substack{R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + 2R_1}} \begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 5 & 7 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & 0 & 2 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$- \begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 7 & 2 \\ 0 & -3 & 0 & 2 \end{vmatrix} \xrightarrow{\substack{R_3 \to R_3 - 5R_2 \\ R_4 \to R_4 + 3R_2}} - \begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -18 \\ 0 & 0 & 6 & 14 \end{vmatrix} \xrightarrow[R_3 \to -\frac{1}{3} R_3]{(\text{"FACTOR OUT" } (-3))}$$

$$-(-3) \begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 6 & 14 \end{vmatrix} \xrightarrow{R_4 \to R_4 - 6R_3} 3 \begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -22 \end{vmatrix}$$

$$= 3(-1)(1)(1)(-22)$$

$$= \boxed{66}$$

<u>NOTE</u>: SINCE $|A| = |A^T|$, WE CAN ACTUALLY DO <u>COLUMN</u> OPERATIONS AS WELL AS ROW OPERATIONS – JUST SWAP ROWS & COLUMNS, DO A ROW OP, AND SWAP BACK. (OF COURSE YOU DON'T NEED TO WRITE ALL OF THAT OUT).

EX: $$\begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & -1 \\ 1 & 3 & 0 \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad (\text{EXPAND ON LAST ROW})$$

IN GENERAL THIS TELLS US:

- IF ANY TWO ROWS OF A ARE THE SAME, $\det A = 0$.

SINCE WE CAN DO COLUMN OPS BY ABOVE:

- IF ANY TWO COLUMNS OF A ARE THE SAME, $\det A = 0$.

MATRIX INVERSES

DEF: IF A IS $n \times n$, WE SAY THAT A IS INVERTABLE OR NONSINGULAR IF THERE IS SOME MATRIX B s.t.

$$AB = BA = I \leftarrow \text{IDENTITY MATRIX}$$

(SO B IS ALSO $n \times n$). B IS CALLED THE INVERSE TO A AND DENOTED A^{-1}. IF SUCH A B DOES NOT EXIST, WE SAY THAT A IS SINGULAR.

PROPERTIES

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

THERE ARE TWO WAYS WE WILL USE TO FIND A^{-1}.

METHOD 1 USE THE FORMULA:

IF A^{-1} EXISTS, $A^{-1} = \frac{1}{\text{DET} A}(\text{ADJ} A)$

WHERE ADJ A (ADJOINT OF A) IS THE $n \times n$ MATRIX:

$$\text{ADJ} A = \begin{pmatrix} C_{11} & C_{12} \dots & C_{1n} \\ C_{21} & C_{22} \dots & C_{2n} \\ \vdots & \vdots \ \ddots & \\ C_{n1} & C_{n2} & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} \dots & C_{n1} \\ C_{12} & C_{22} \dots & C_{n2} \\ \vdots & \vdots \ \ddots & \\ C_{1n} & C_{2n} & C_{nn} \end{pmatrix} \quad \left(C_{ij} \text{ COFACTORS OF } A \right)$$

IN PARTICULAR, THIS TELLS US:

① IF A^{-1} EXISTS, IT IS UNIQUE (THERE IS A FORMULA!)

② A^{-1} EXISTS IFF $\text{DET}(A) \neq 0$. IF A^{-1} EXISTS, $\text{DET}(A) \neq 0$ SINCE WE ARE DIVIDING BY IT IN THE FORMULA ABOVE. IF $\text{DET}(A) \neq 0$, A^{-1} EXISTS JUST BY USING THE FORMULA.

EX: $A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$ FIND A^{-1}

FIRST WE FIND THE MINORS M_{ij}:

$$M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \boxed{M_{12} = \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -5} \qquad M_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$\boxed{M_{21} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2} \qquad M_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \qquad \boxed{M_{23} = \begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = -6}$$

$$M_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad \boxed{M_{32} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2} \qquad M_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6$$

THE COFACTORS ARE THE SAME BUT PUT A MINUS IN FRONT OF THE CIRCLED M's:

$$\text{ADJ } A = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & -3 \\ -2 & 2 & 6 \\ 2 & -2 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$$

$$\begin{vmatrix} \boxed{2 \quad 2 \quad 0} \\ -2 \quad 1 \quad 1 \\ 3 \quad 0 \quad 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 2 - 2(-5)$$

$$= 12 = \text{DET}(A)$$

SO $A^{-1} = \frac{1}{\text{DET} A} \text{ ADJ} A = \boxed{\frac{1}{12} \begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix}}$

NOW WE DO ANOTHER METHOD WHICH IS A LITTLE LESS TEDIOUS. FIRST A REMARK ON INVERTABILITY AND RANK. SUPPOSE A IS $n \times n$:

CLAIM: $\text{DET}(A) \neq 0$ IFF $\text{RANK}(A) = n$.

WE CAN APPLY ROW OPS TO $|A|$ TO GET IT IN ITS ROW-ECHELON FORM. EACH OPERATION MULTIPLIES $|A|$ BY SOME NONZERO CONSTANT. SO WE HAVE SOMETHING LIKE: (SAY A IS 3×3)

$$\text{DET}(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \underset{\xrightarrow[\text{ROW OPS}]{}}{=} c \begin{vmatrix} A \text{ IN} \\ \text{ROW-ECHELON FORM} \end{vmatrix} = C \begin{vmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{vmatrix} \text{ IF } A \text{ IS RANK } 3$$

$* \text{ MEANS "SOMETHING"}$

$$= C \neq 0$$

BUT IF RANK$(A) = 0, 1,$ OR 2, THERE WILL BE A ROW OF ZEROS IN ITS ROW-ECHELON FORM. AND SO DET$(A) = 0$.

SO A RANK 3 $\Longrightarrow$ DET$(A) \neq 0$ SINCE DET$(A) = C$

AND DET$(A) \neq 0 \Longrightarrow A$ RANK 3 SINCE OTHERWISE ITS ROW-ECHELON FORM HAS A ZERO ROW AND THEN DET$(A) = c$ DET$(A$ IN R.E.F.$) = 0$

SO WE CAN SAY:

LET A BE $n \times n$. THE FOLLOWING ARE EQUIVALENT

$$\left.\begin{array}{l} ① \ A^{-1} \text{ EXISTS} \\ ② \ \text{DET} A \neq 0 \\ ③ \ RK(A) = n \end{array}\right\} \text{VERY USEFUL}$$

NOW FOR:

METHOD 2

LET A BE $n \times n$. TO FIND ITS INVERSE:

- PUT A INTO AN AUGMENTED MATRIX WITH AN $n \times n$ IDENTITY MATRIX

$$(A \mid I)$$

- ROW REDUCE THE MATRIX INTO THE FORM:

$$(I \mid B)$$

  i.e. GET THE LEFT HALF TO BE I. THEN $B = A^{-1}$

- WE CAN ROW REDUCE A TO BECOME I IFF ITS ROW-ECHELON FORM HAS n NONZERO ROWS - i.e. IFF IT IS RANK n - i.e. IFF A^{-1} EXISTS BY ABOVE. SO, IF WE CAN'T GET THE LEFT BLOCK TO BE I, A^{-1} DOESN'T EXIST.

EX: $A = \begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}$ FIND A^{-1}

$$(A|I) = \left(\begin{array}{cc|cc} 6 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{6}R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & 0 \\ -3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 + 3R_1}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & 0 \\ 0 & 2 & \frac{1}{2} & 1 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \end{array}\right)$$

$$\underbrace{\begin{matrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{matrix}}_{A^{-1}}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

CHECK: $\begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

NOTE: 2×2 MATRICES HAVE SIMPLE INVERSES:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ FOR AN INVERTABLE 2×2 MATRIX

THIS FOLLOWS FROM METHOD 1 IF WE USE THE DEFINITION THAT DET (A) FOR A 1×1 MATRIX IS JUST THE NUMBER ITSELF:

MINORS OF $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $M_{11} = |d| = d$ $\boxed{M_{12} = |c| = c}$

$\boxed{M_{21} = |c| = b}$ $M_{22} = |d| = a$

} HERE $|d|$ MEANS DETERMINANT OF THE 1×1 MATRIX (d)

CIRCLED M's FLIP SIGN TO GO TO C's (COFACTERS): $C_{11} = d$ $C_{12} = -c$

$C_{21} = -b$ $C_{22} = a$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ADJ}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ BY METHOD 1

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$