LAST TIME:

- WE TALKED ABOUT HOW:

$$\boxed{RK(A) = \#\text{ L.I. ROWS} = \#\text{ L.I. COLUMNS}}$$

IN PARTICULAR THIS GAVE US A WAY TO TELL IF n VECTORS $V_1, \ldots, V_n$ ARE L.I.: PUT THEM AS THE ROWS OR COLUMNS OF A MATRIX AND THE RANK TELLS US HOW MANY OF THE V's ARE L.I. SO $\{V_1, \ldots, V_n\}$ ARE L.I. IFF THE MATRIX IS RANK n.

- WE DEFINED THE DETERMINANT FOR $n \times n$ MATRICES AND SAW HOW IT BEHAVED UNDER ROW OPERATIONS
- WE DEFINED A^{-1} FOR AN $n \times n$ MATRIX A, AND FOUND IT TWO DIFFERENT WAYS:

① $A^{-1} = \frac{1}{\det A} \operatorname{ADJ} A$

② $(A \mid I) \xrightarrow[\text{REDUCE}]{\text{ROW}} (I \mid A^{-1})$

- WE SAW A <u>VERY IMPORTANT</u> STRING OF EQUIVALENT STATEMENTS:

☺ A^{-1} EXISTS $\iff \det A \neq 0 \iff RK(A) = n$

(WHERE A IS $n \times n$)

LAST TIME WE FOUND A^{-1} OF $A = \begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}$: $\quad A^{-1} = \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

THE UTILITY OF INVERSES IS THIS: SUPPOSE WE WANT TO SOLVE THE SYSTEM:

$$\begin{aligned} 6x &= 2 \\ -3x + 2y &= 1 \end{aligned} \iff \begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

JUST MULTIPLY BOTH SIDES BY A^{-1}:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}}_{A^{-1}A = I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$$

SO $x = \frac{1}{3}$, $y = 1$

IN GENERAL, WHEN SOLVING $Ax = b$ WITH SOME $n \times n$ A WHICH IS INVERTABLE, WE CAN FIND x BY JUST MULTIPLYING BY A^{-1} AS ABOVE.

(NOTE: x AND b ARE <u>VECTORS</u> HERE)

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

HOMOGENEOUS SYSTEMS

RECALL THAT A HOMOGENEOUS SYSTEM OF EQ'S IS OF THE FORM $Ax = 0$.
SINCE $x=0$ IS ALWAYS A SOLUTION, CALLED THE TRIVIAL SOLUTION, THE SYSTEM IS ALWAYS CONSISTENT.

SUPPOSE A IS 3×3 AND WE WANT TO SOLVE $Ax=0$. WE USE AN AUGMENTED MATRIX TO FIND SOL'S:

CASE 1 $\operatorname{RK}(A)=3$ THEN $\left(A \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right) \xrightarrow{\text{ROW OPS}} \left(\begin{matrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{matrix} \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right)$ FOR SOME $a,b,c \in \mathbb{R}$

THEN: E_3: $z=0$

E_2: $y + \cancel{cz} = y = 0$

E_1: $x + \cancel{ay} + \cancel{bz} = x = 0$

SO $\operatorname{RK}(A)=3 \implies Ax=0$ HAS ONLY TRIVIAL SOLUTION

CASE 2 $\operatorname{RK}(A)<3$ THEN $\left(A \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right) \xrightarrow{\text{ROW OPS}} \left(\begin{matrix}1 & a & b\\0 & 1 & c\\0 & 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right)$ OR $\left(\begin{matrix}1 & a & b\\0 & 0 & 0\\0 & 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right)$ OR ... (ZERO ROW SOMEWHERE)

IN ANY OF THESE CASES, WE HAVE A PARAMETER:

$$\#\text{PARAMS} = \#\text{VARS} - \operatorname{RK}(A) = 3 - \operatorname{RK}(A) > 0$$

THUS ONE OF x, y OR z WILL BE A FREE VARIABLE WE CAN CHOOSE TO BE NONZERO.

SO, $\operatorname{RK}(A) < 3 \implies Ax=0$ HAS NONTRIVIAL SOLUTIONS

FOR AN $n\times n$ MATRIX WE HAVE (BY THE SAME REASONING)

$Ax=0$ HAS NONTRIVIAL SOLUTIONS $\iff \operatorname{RK}(A) < n$

THE IDEA IS THAT AS LONG AS WE HAVE A PARAMETER, WE CAN CHOOSE A NONTRIVIAL SOLUTION. BUT $\#\text{PARAMS} = n - \operatorname{RK}(A)$
SO WE HAVE PARAMETERS IFF $\operatorname{RK}(A) < n$.

- NOW SINCE $\operatorname{RK}(A)$ CAN ONLY BE $1, 2, 3, \dots, n-1$, OR n, $\operatorname{RK}(A) < n$ IS EQUIVALENT TO $\operatorname{RK}(A) \neq n$. BUT BY ☺ $\operatorname{RK}(A) \neq n \iff \boxed{\det(A) = 0} \iff A^{-1}$ DOES NOT EXIST

(WE JUST NEGATED ALL THE STATEMENTS IN ☺)

ONE WE WANT

IN OTHER WORDS:

$Ax=0$ HAS NONTRIVIAL SOLUTIONS $\iff \det(A) = 0$

THIS IS WHAT WE'LL NEED TO FIND EIGENVECTORS / VALUES

EIGENVECTORS

DEF: LET A BE AN $n \times n$ MATRIX. WE SAY THAT A VECTOR $v \in \mathbb{R}^n$ IS AN <u>EIGENVECTOR</u> WITH <u>EIGENVALUE</u> $\lambda \in \mathbb{R}$ IF:

① $v \neq 0$ (<u>NOT</u> ZERO VECTOR)

② $Av = \lambda v$

(IF WE DIDN'T RULE OUT THIS CASE, THE ZERO VECTOR WOULD BE AN EIGENVECTOR OF <u>ANY</u> MATRIX WITH EIGENVALUE 0)

SO EIGENVECTORS OF SOME MATRIX ARE VECTORS THAT JUST SCALE BY λ WHEN MULTIPLIED BY A.

EX: $A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$ THEN $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ IS AN EIGENVECTOR WITH WHAT EIGENVALUE?

$$Av = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2v$$

SO $\lambda = 2$

HOW DO WE FIND THESE VECTORS? WELL, FIRST WE FIND THE λ's.

SO, WE DESIRE A $\lambda \in \mathbb{R}$ SUCH THAT:

$\{\lambda$ SUCH THAT: $Av = \lambda v$ FOR SOME $v \neq 0\}$

$= \{\lambda$ SUCH THAT: $Av = \lambda I v$ FOR SOME $v \neq 0\}$ (I IS $n \times n$ IDENTITY MATRIX)

$= \{\lambda$ SUCH THAT: $(A - \lambda I)v = 0$ FOR SOME $v \neq 0\}$ (FACTOR OUT v)

$= \{\lambda$ SUCH THAT: $(A-\lambda I)v = 0$ HAS NONTRIVIAL SOL'S$\}$

$= \{\lambda$ SUCH THAT: THE HOMOG. SYSTEM $\det(A - \lambda I) = 0\}$ BY WHAT WE SAID BEFORE

$= \{\lambda$ SUCH THAT: $\det(A - \lambda I) = 0\}$

EX: FIND THE EIGENVALUES OF $A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 2$$
$$= \lambda^2 - 3\lambda + 2 - 2$$
$$= \lambda(\lambda - 3) = 0$$

$\lambda = 0$ & $\lambda = 3$

NOW ONCE WE HAVE THE λ'S, WE CAN FIND THE EIGENVECTORS CORRESPONDING TO THEM:

$$Av = \lambda v \iff \underset{\substack{\uparrow \\ \text{SOLVE THIS SYSTEM} \\ \text{OF EQ'S FOR EACH } \lambda}}{(A - \lambda I)v = 0}$$

SO CONTINUING WITH OUR ABOVE EXAMPLE:

$\lambda_1 = 0$: SOLVE $(A - 0I)v = 0$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$2x + y = 0$

$x = -\frac{1}{2}y$

$\begin{pmatrix} -\frac{1}{2}y \\ y \end{pmatrix}$ ARE ALL EIGENVECTORS

CHOOSE SAY $y = 2$:

$$v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

λ_2: $(A - 3I)v = 0$

$$\left(\begin{array}{cc|c} 2-3 & 1 & 0 \\ 2 & 1-3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-x + y = 0$

$x = y$

$\begin{pmatrix} y \\ y \end{pmatrix}$ ARE ALL EIGENVECTORS

CHOOSE $y = 1$:

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

NOTICE:

- OUR CHOICE OF EIGENVECTORS IS HARDLY UNIQUE – SCALING AN EIGENVECTOR STILL GIVENS YOU AN EIGENVECTOR
- OUR SYSTEMS $(A - \lambda I)v = 0$ BOTH HAD PARAMETERS. IF THIS DOES NOT HAPPEN, THEN YOU EITHER ROW-REDUCED INCORRECTLY OR FOUND THE WRONG λ'S.

EX: FIND THE EIGENVALUES/VECTORS OF $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

FIND λ'S: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \quad \lambda = 2, 2$

FIND v'S: $\lambda = 2$: $(A - \lambda I)v = 0$

$$\left(\begin{array}{cc|c} 2-2 & 0 & 0 \\ 0 & 2-2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

NO RESTRICTIONS ON x AND y SO ANY VECTOR $\begin{pmatrix} x \\ y \end{pmatrix}$ IS AN EIGENVECTOR OF A. NOTICE $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

IN PARTICULAR NOTE THAT WE CAN CHOOSE 2 L.I. EIGENVECTORS CORRESPONDING TO $\lambda=2$, SAY $V_1=\begin{pmatrix}1\\0\end{pmatrix}$, $V_2=\begin{pmatrix}0\\1\end{pmatrix}$. DOES THIS HAVE SOMETHING TO DO WITH $\lambda=2$ OCCURRING TWICE (MULTIPLICITY TWO)? <u>YES</u>

THM: SUPPOSE λ_1 IS AN EIGENVALUE OF A $(n\times n)$ WITH MULTIPLICITY m (i.e. THE FACTOR $(\lambda-\lambda_1)$ OCCURS m TIMES IN $\det(A-\lambda I)=0$). THEN THE # OF L.I. EIGENVECTORS WITH EIGENVALUE λ_1 IS <u>AT MOST</u> m (BUT NOT ALWAYS EQUAL TO m)

EX: FIND ALL OF THE EIGENVALUES/VECTORS OF $A=\begin{pmatrix}2&1\\0&2\end{pmatrix}$

FIND λ's: $\det(A-\lambda I)=\det\begin{pmatrix}2-\lambda&1\\0&2-\lambda\end{pmatrix}=(2-\lambda)^2=0$ $\quad\lambda=2,2$

FIND V's: $(A-2I)V=0\longrightarrow\left(\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right)$ SO $\underline{Y=0}$

OUR SOLUTIONS ARE $\begin{pmatrix}X\\0\end{pmatrix}$ (ONLY 1 PARAMETER)

SO WE ONLY HAVE ONE L.I. EIGENVECTOR SAY $V=\begin{pmatrix}1\\0\end{pmatrix}$

NOTE: WHEN SOLVING $(A-\lambda I)V=0$, # PARAMS = # OF L.I. EIGENVECTORS

RECALL THAT WE CHOSE THE λ's SO THAT WE WOULD ALWAYS HAVE PARAMETERS.

THUS:

- TO ANY EIGENVALUE λ, THERE IS <u>AT LEAST</u> ONE EIGENVECTOR

EX: FIND THE EIGENVALUES/VECTORS OF $A=\begin{pmatrix}1&1&0\\-4&-3&0\\11&16&2\end{pmatrix}$

FIND λ's: $\det(A-\lambda I)=\begin{vmatrix}1-\lambda&1&0\\-4&-3-\lambda&0\\11&16&2-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}1-\lambda&1\\-4&-3-\lambda\end{vmatrix}$

$=(2-\lambda)(\lambda^2+2\lambda+1)$

$=(2-\lambda)(\lambda+1)^2=0$

$\lambda=2,-1,-1$

$\lambda_1=2$:

FIND V's: $(A-2I)V=0$

$$\left(\begin{array}{ccc|c}-1&1&0&0\\-4&-5&0&0\\11&16&0&0\end{array}\right)\xrightarrow[R_3\to R_3+11R_1]{R_2\to R_2-4R_1}\left(\begin{array}{ccc|c}-1&1&0&0\\0&-9&0&0\\0&27&0&0\end{array}\right)\xrightarrow{R_3\to R_3+3R_2}\left(\begin{array}{ccc|c}-1&1&0&0\\0&-9&0&0\\0&0&0&0\end{array}\right)$$

$R_1 \to -R_1$
$R_2 \to -\frac{1}{9}R_2$
$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$E_2: \ y = 0$

$E_1: \ x - y = x = 0$ CHOOSE $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

1 PARAMETER = 1 E. VECTOR

$\lambda_2 = -1$:

$$(A + I)v = 0 \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -4 & -2 & 0 & 0 \\ 11 & 16 & 3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - \frac{11}{2}R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{21}{2} & 3 & 0 \end{array}\right) \xrightarrow[R_3 \to \frac{2}{21}R_3]{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$E_2: \ y + \frac{2}{7}z = 0$

$y = -\frac{2}{7}z$

$E_1: \ x + \frac{1}{2}y = 0$

1 PARAM = 1 E. VECTOR

$x = -\frac{1}{2}y = -\frac{1}{2}\left(-\frac{2}{7}z\right) = \frac{1}{7}z$

SOL: $\begin{pmatrix} \frac{1}{7}z \\ -\frac{2}{7}z \\ z \end{pmatrix}$ SO LET $z = 7$: $V_2 = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$

AGAIN WE ONLY HAD 1 E.VECTOR CORRESPONDING TO $\lambda = -1$ (MULTIPLICITY 2)

- COMPLEX λ'S:

THE EIGENVALUES CAN BE COMPLEX NUMBERS. IN THIS CASE, IF A IS A <u>REAL</u> $n \times n$ MATRIX, EIGENVALUES/VECTORS COME IN <u>CONJUGATE PAIRS</u>.

DEF: FOR A NUMBER $a + ib \in \mathbb{C}$ (COMPLEX #'S), THE <u>COMPLEX CONJUGATE</u>, DENOTED $\overline{a+ib}$, IS JUST $\overline{a+ib} = a - ib$.

- FOR ANY COMPLEX NUMBERS $z_1 = a + ib$ AND $z_2 = c + id$, $\overline{z_1}\,\overline{z_2} = \overline{z_1 z_2}$ AND $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$

SO THE "BAR" DISTRIBUTES OVER + & −.

- FOR A MATRIX A, $\overline{A}$ = A CONJUGATED AT EACH ELEMENT:

EX: $\overline{\begin{pmatrix} i & 3 \\ 1-i & 4 \end{pmatrix}} = \begin{pmatrix} -i & 3 \\ 1+i & 4 \end{pmatrix}$

NOW SUPPOSE WE HAVE A $n\times n$ (REAL) WITH EIGENVALUE/VECTOR λ AND v.

THEN: $Av = \lambda v$

TAKE CONJUGATE: $\overline{Av} = \overline{\lambda v}$

$$\bar{A}\,\bar{v} = \bar{\lambda}\,\bar{v}$$

IS BECAUSE A IS A REAL MATRIX SO THERE ARE NO i'S TO FLIP

$$A\bar{v} = \bar{\lambda}\,\bar{v}$$

IN OTHER WORDS, IF v IS A COMPLEX EIGENVECTOR OF A REAL MATRIX A WITH COMPLEX EIGENVALUE λ, THEN $\bar{v}$ IS ANOTHER EIGENVECTOR OF A WITH EIGENVALUE $\bar{\lambda}$.

EX: FIND EIGENVALUES/VECTORS OF $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

FIND λ's: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2+1 = 0$

$(2-\lambda)^2 = -1$ ... TAKE $\sqrt{\ }$

$2-\lambda = \pm i$

$2 \pm i = \lambda$

SO $\lambda_1 = 2+i \quad \lambda_2 = 2-i$

$\lambda_1 = \bar{\lambda}_2$

FIND v's: $(A-(2+i)I)v = 0$

$\lambda_1 = 2+i$:

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + iR_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right)$$

1 PARAM

$x - iy = 0$

$x = iy$

$\begin{pmatrix} iy \\ y \end{pmatrix}$ ARE ALL E. VECTORS CHOOSE $y=1$

$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

BY ABOVE, WE KNOW WE CAN LET

$v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \bar{v}_1$ BE AN EIGENVECTOR FOR λ_2

BUT LET'S JUST CHECK THAT:

$\lambda_2 = 2-i$: $(A-(2-i)I)v = 0$

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - iR_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right)$$

$x + iy = 0$

$x = -iy$

$\begin{pmatrix} -iy \\ y \end{pmatrix}$ ALL E. VECTORS CHOOSE $y=1$

$v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

RMK: SINCE DET (A) FOR AN UPPER OR LOWER TRIANGULAR MATRIX IS THE PRODUCT ALONG THE DIAGONAL, WE HAVE: THAT FOR ANY UPPER OR LOWER TRIANGULAR MATRIX, THE EIGENVALUES ARE THE DIAGONAL ELEMENTS.

EX: $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$ $\quad \det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 1 & 0 & 4-\lambda & 0 \\ 0 & 0 & 2 & -1-\lambda \end{vmatrix} = -\lambda(3-\lambda)(4-\lambda)(-1-\lambda)$

SO $\lambda = 0, 3, 4, -1$

DIAGONALIZATION

DEF: AN $n \times n$ MATRIX A IS SAID TO BE DIAGONALIZABLE IF THERE EXISTS SOME MATRIX P THAT IS INVERTABLE AND A DIAGONAL MATRIX D S.T.

$$PD = AP$$

"PRETTY DOGS = AWESOME PETS"

HOW DO WE KNOW WHEN WE CAN DO THIS? HERE IS WHEN:

THM: AN $n \times n$ MATRIX A IS DIAGONALIZABLE IFF IT HAS n L.I. EIGENVECTORS. THEN, SUPPOSE A HAS EIGENVECTORS $v_1, \ldots, v_n$ AND VALUES $\lambda_1, \ldots, \lambda_n$, THEN:

P = MATRIX WITH COLUMNS $\left(v_1 \middle| v_2 \middle| \ldots \middle| v_n \right)$

D = DIAGONAL WITH ELEMENTS $\lambda_1 \ldots \lambda_n$: $\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$

ONE THING THAT IS TRUE:

THM: FOR ANY $n \times n$ MATRIX A, THE EIGENVECTORS CORRESPONDING TO DIFFERENT λ'S ARE ALL L.I.

PF: SUPPOSE $v_1 \sim \lambda_1, v_2 \sim \lambda_2, \ldots, v_k \sim \lambda_k$ ARE ALL EIGENVECTORS/VALUES OF A, WITH ALL λ'S BEING DISTINCT. WE SUPPOSE $c_1 v_1 + \ldots + c_k v_k = 0$.

WE WANT TO SHOW THAT ALL THE C'S $= 0$

WE PROVE BY INDUCTION ON k.

IF $k=2$:

☆ $c_1 v_1 + c_2 v_2 = 0$ $\quad \lambda_1 \neq \lambda_2$

SO $c_1 A v_1 + c_2 A v_2 = 0$ (MULT ☆ BY A) AND $c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0$ (MULT ☆ BY λ_1)

$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$

SUBTRACTING THESE TWO EQ'S $\quad c_2(\lambda_1 - \lambda_2) v_2 = 0$

SO $c_2 = 0$ OR $\underbrace{\lambda_1 = \lambda_2}_{\text{NOT TRUE}}$ (v_2 IS NOT A ZERO VECTOR BY DEF. OF AN E. VECTOR)

SO $c_2 = 0$. THUS $c_1 = 0$ AS WELL, AND WE HAVE PROVEN THE CASE $k=2$.

NOW FOR THE GENERAL CASE:

☽ $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$ $\quad \lambda$'s DISTINCT

SO $c_1 A v_1 + c_2 A v_2 + \dots + c_k A v_k = 0$ AND $c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 + \dots + c_k \lambda_1 v_k = 0$ (MULT EQ ☽ BY λ_1)

$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_k \lambda_k v_k = 0$

SUBTRACTING THESE TWO EQ'S: ($c_1 \lambda_1 v_1$ CANCELS)

$$c_2(\lambda_1 - \lambda_2) v_2 + c_3(\lambda_1 - \lambda_3) v_3 + \dots + c_k(\lambda_1 - \lambda_k) v_k = 0$$

BUT BY INDUCTION, THE $k-1$ EIGENVECTORS $v_2 \dots v_k$ ARE L.I., THUS

$$c_2(\lambda_1 - \lambda_2) = 0, \quad c_3(\lambda_1 - \lambda_3) = 0, \quad \dots \quad c_k(\lambda_1 - \lambda_k) = 0$$

BUT THEN BY DISTINCTNESS OF THE λ's, $c_2 = c_3 = \dots = c_k = 0$, AND THUS SO ALSO DOES c_1. THIS PROVES THE GENERAL CASE ∎

BY THIS THM, FIND OUT IF WE CAN DIAGONALIZE A CAN BE REDUCED TO CHECKING L.I. OF EIGENVECTORS FOR EACH λ SEPARATELY, NAMELY:

◎ TO SEE IF YOU CAN DIAGONALIZE A YOU ☆

1. FIND ALL OF THE EIGENVALUES
2. FOR EACH λ OF MULTIPLICITY m, SEE IF YOU CAN FIND m L.I. EIGENVECTORS CORRESPONDING TO λ.
3. IF YOU CAN FIND SUCH VECTORS IN (2) FOR <u>ALL</u> λ's, A IS DIAGONALIZABLE. IF NOT, A IS NOT DIAGONALIZABLE.

THIS IS ABOUT ALL YOU WILL NEED TO KNOW

EX: CAN WE DIAGONALIZE $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$?

FIND λ's: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)$ $\qquad \lambda = 2, 3$

TO EACH λ, WE HAVE AT LEAST ONE E. VECTOR. THE V'S FOR $\lambda = 2$ AND $\lambda = 3$ WILL BE L.I. BY OUR THM, SO <u>YES</u> WE CAN DIAGONALIZE. LET'S CHECK:

FIND V'S:

$\lambda_1 = 2$: $(A - 2I)v = 0$

$$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \longrightarrow y = 0 \text{ so } v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 3$: $(A - 3I)v = 0$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow x = y \text{ so } v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

SO BY OUR THM WE CAN LET $P = \left(v_1 \mid v_2\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ AND $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

THEN $PD = AP$

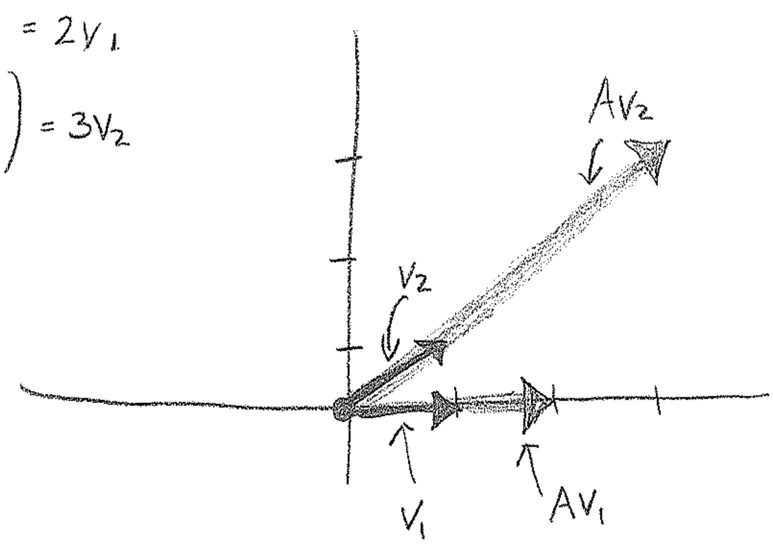
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix} \checkmark$$

GEOMETRICALLY, WHAT IS GOING ON? A IS JUST SCALING THESE TWO VECTORS.

$$Av_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2v_1$$

$$Av_2 = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3v_2$$



SINCE V_1 AND V_2 ARE L.I., EVERY VECTOR IN $\mathbb{R}^2$ IS A LINEAR COMBINATION OF THEM. SINCE A ACTS VERY SIMPLY ON THESE VECTORS, WE CAN ESSENTIALLY CHANGE MATRIX MULTIPLICATION TO SCALING BY THE λ's.

EX: $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

THEN
$$A\begin{pmatrix}4\\2\end{pmatrix} = A\left(2\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}1\\0\end{pmatrix}\right)$$
$$= 2A\begin{pmatrix}1\\1\end{pmatrix} + 2A\begin{pmatrix}1\\0\end{pmatrix}$$
USING $Av = \lambda v$
$$= 2(3)\begin{pmatrix}1\\1\end{pmatrix} + 2(2)\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix}6\\6\end{pmatrix} + \begin{pmatrix}4\\0\end{pmatrix}$$
$$= \begin{pmatrix}10\\6\end{pmatrix}$$

WE CAN REWRITE $PD = AP$ AS $PDP^{-1} = A$

NOTICE THAT IN OUR EXAMPLE $P\begin{pmatrix}1\\0\end{pmatrix} = V_1$, $P\begin{pmatrix}0\\1\end{pmatrix} = V_2$

SO $\begin{pmatrix}1\\0\end{pmatrix} = P^{-1}V_1$, $\begin{pmatrix}0\\1\end{pmatrix} = P^{-1}V_2$

SO IF WE THINK OF MULTIPLICATION BY A IS MULT. BY PDP^{-1},

$$AV_1 = PDP^{-1}V_1 = PD\begin{pmatrix}1\\0\end{pmatrix} = P\,2\begin{pmatrix}1\\0\end{pmatrix} = 2P\begin{pmatrix}1\\0\end{pmatrix} = 2V_1$$

IN OTHER WORDS,
- P^{-1} TAKES THE EIGENVECTORS TO THE STANDARD BASIS VECTORS
- D SCALES THEM ACCORDING TO THE CORRECT λ
- P PUTS EVERYTHING BACK IN TERMS OF THE EIGENVECTORS

FOR THIS REASON, WE CALL P THE "CHANGE OF BASIS" MATRIX CORRESPONDING TO CHANGING BETWEEN THE STANDARD BASIS AND THE BASIS OF EIGENVECTORS. THE POINT IS THAT IF YOU CHOOSE THE BASIS OF EIGENVECTORS AND PUT ALL OF THE VECTORS IN TERMS OF THEM (AS ABOVE) MULT. BY A IS VERY SIMILAR TO MULT. BY D.

EX: $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ CAN YOU DIAGONALIZE A?

λ's: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$

$$= (2-\lambda)^2 \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(\lambda^2-1) = 0$$

$$\lambda = 2, 2, \pm 1$$

LET $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ (MULT 2)

FOR λ_1 & λ_2 WE WILL HAVE EXACTLY ONE L.I. E.VECTOR. THIS IS BECAUSE THERE IS ALWAYS AT LEAST ONE, AND AT MOST m WHERE m IS THE MULTIPLICITY OF THE λ (WHICH IS 1). λ_3 COULD HAVE 1 OR 2 L.I. E.VECTORS. IF IT IS 2, WE CAN DIAGONALIZE. SO WE CHECK:

FOR λ_3:

$(A-2I)v = 0$ CORRESPONDS TO:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + 2R_4} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right)$$

SINCE WE HAVE 4 VARIABLES WE USE x, y, z & w.

E_3: $-3w = 0$ SO $w = 0$

E_4: $\cancel{w} - 2z = -2z = 0$ SO $z = 0$

OUR EIGENVECTORS FOR $\lambda = 2$ ARE:

$\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix}$ FOR ANY x, y SO CHOOSE

$V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ BOTH L.I.

THUS WE CAN DIAGONALIZE

LET'S FIND P & D TO COMPLETE THE EXAMPLE

WE NEED THE EIGENVECTORS TO λ_1 & λ_2:

λ_1: $(A-(1)I)V=0$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_4 \to R_4 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

E_1: $X=0$
E_2: $Y=0$
E_3: $-Z+W=0$ SO $Z=W$

$\begin{pmatrix} 0 \\ 0 \\ Z \\ Z \end{pmatrix}$ ALL E. VECTORS CHOOSE $Z=1$

$$V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

λ_2: $(A-(-1)I)V$

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_4 \to R_4 - R_3,\ R_1 \to \frac{1}{3}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

E_1: $X=0$
E_2: $Y=0$
E_3: $Z+W=0$ SO $Z=-W$

$\begin{pmatrix} 0 \\ 0 \\ Z \\ -Z \end{pmatrix}$ ARE ALL E. VECTORS CHOOSE $Z=1$

$$V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$P = \left(V_1 \,\middle|\, V_2 \,\middle|\, V_3 \,\middle|\, V_4\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

AND

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

THE ORDER IS VERY IMPORTANT. NOTICE HOW THE 1ST COLUMN OF P (WHICH IS V_1) HAS E. VALUE λ_1 (1ST COLUMN OF D) AND THE 2ND COLUMN OF P (WHICH IS V_2) HAS E. VALUE λ_2 (2ND COLUMN OF D) ETC.

CHECK THAT $PD=AP$