FIRST A FEW USEFUL FACTS IN LINEAR ALGEBRA

(1) A SYMMETRIC MATRIX $(A^T = A)$ IS ALWAYS DIAGONALIZABLE & HAS REAL λ's

(2) IF A HAS EIGENVALUES $\lambda_1 \dots \lambda_n$ THEN

$\boxed{\text{DET}(A) = \lambda_1 \lambda_2 \dots \lambda_n}$ (PRODUCT OF EIGENVALUES)

IF A IS DIAGONALIZABLE, $A = PDP^{-1}$ SO

$$\text{DET}(A) = \cancel{\text{DET}(P)}\, \text{DET}(D)\, \cancel{\text{DET}(P^{-1})} \xrightarrow{\text{SINCE}} \text{DET}(PP^{-1}) = \text{DET}(P)\,\text{DET}(P^{-1})$$

$$\text{DET}(I) = \text{DET}(P)\,\text{DET}(P^{-1})$$

$$1 = \text{DET}(P)\,\text{DET}(P^{-1})$$

$$= \text{DET}(D)$$

$$= \lambda_1 \dots \lambda_n$$

IF A IS NOT DIAGONALIZABLE, THERE IS STILL SOME MATRIX P s.t. $A = PJP^{-1}$ WHERE J IS UPPER-TRIANGULAR WITH THE λ's OF A ALONG THE DIAGONAL (SO THE SAME PROOF WORKS).

(3) THE TRACE OF AN $n \times n$ MATRIX A IS THE SUM OF THE ENTRIES ON THE MAIN DIAGONAL: $\text{TR}\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} = 1 + 4 = 5$ $\quad \text{TR}\begin{pmatrix} -1 & 2 & 2 \\ 1 & 7 & 2 \\ 0 & 0 & \pi \end{pmatrix} = -1 + 7 + \pi = \pi + 6$

IF A HAS EIGENVALUES $\lambda_1 \dots \lambda_n$ THEN

$\boxed{\text{TR}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n}$

THESE FACTS (2) AND (3) ARE USEFUL FOR QUICK CHECKS ON IF YOU FOUND THE CORRECT λ's.

EX: SUPPOSE YOU FOUND THE λ's OF $A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$ TO BE $\lambda_1 = 5$ $\lambda_2 = 5$

$\text{TR}(A) = 3 + 7 = 10$

$\text{DET}(A) = 21 + 4 = 25$

SO $\text{TR}(A) = \lambda_1 + \lambda_2$ AND $\text{DET}(A) = \lambda_1 \lambda_2$, SO WE ARE PROBABLY RIGHT.

NOTE: IN THE 2×2 CASE, $\text{DET}(A)$ AND $\text{TR}(A)$ COMPLETELY DETERMINE λ_1 AND λ_2. SO IF YOUR λ's SATISFY $\lambda_1 + \lambda_2 = \text{TR}(A)$, $\lambda_1 \lambda_2 = \text{DET}(A)$ THEN YOUR λ's ARE CORRECT.

(4) AN ORTHOGONAL MATRIX A IS A MATRIX s.t. $A^T = A^{-1}$. JUST BE FAMILIAR WITH THE DEFINITION. THESE MATRICES PRESERVE ANGLES BETWEEN VECTORS IN $\mathbb{R}^n$, i.e. $Av \cdot Aw = v \cdot w$ (DOT PRODUCTS)

LAST TIME: DIAGONALIZATION & QUIZZO

A CAN BE DIAGONALIZED IFF IT HAS n L.I. E. VECTORS (IF A IS $n \times n$)

THEN $PD = AP$ WHERE $P = (v_1 | v_2 | \dots | v_n)$ $\quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

EIGENVECTORS OF A W/ EIGENVALUES $\lambda_1, \dots, \lambda_n$

NOW WE MOVE ON TO DIFFERENTIAL EQUATIONS

DEF: AN n-TH ORDER INITIAL VALUE PROBLEM IS GIVEN SOME FUNCTIONS $g(x), a_0(x) \dots a_n(x)$, WE WANT A FUNCTION $y(x)$ s.t.

$$a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \dots + a_1(x)\, y'(x) + a_0(x)\, y(x) = g(x)$$

SUBJECT TO THE INITIAL VALUES:

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots \quad, y^{(n-1)}(x_0) = y_{n-1}$$

NOTE: $y^{(n)}(x)$ MEANS n-TH DERIVATIVE

THM: IF $a_0(x), \dots, a_n(x), g(x)$ ARE CONTINUOUS AND $a_n(x) \neq 0$ ON SOME INTERVAL $I = (b, c) = \{x \mid b < x < c\}$, AND $x_0 \in I$ THEN

① A SOLUTION EXISTS TO THE IVP

② THE SOLUTION IS UNIQUE (WE WILL NEED THIS)

DEF: IF $g(x) = 0$, THEN THE IVP IS SAID TO BE HOMOGENEOUS.
IF NOT, THE IVP IS SAID TO BE NONHOMOGENEOUS.

IF WE GIVE OTHER INITIAL CONDITIONS, WE CAN HAVE NO, ONE, OR MANY SOLUTIONS. A GOOD EXAMPLE (STOLEN FROM THE BOOK) OF THIS IS:

EX: SOLVE $y'' + 16y = 0$

IT TURNS OUT THAT ANY SOLUTION OF THIS EQ. IS OF THE FORM:

$$y = c_1 \cos 4x + c_2 \sin 4x$$

SUPPOSE OUR INITIAL CONDITIONS WERE:

$y(0) = 0,\ y(\frac{\pi}{2}) = 0$ SO $y(0) = c_1 \cos 0 + c_2 \sin 0 = \underline{c_1 = 0}$

$y(\frac{\pi}{2}) = c_2 \underbrace{\sin \tfrac{\pi}{2}(4)}_{=0} = 0$ FOR ANY c_2

SO WE GET $y = c_2 \sin 4x$ FOR ANY c_2 ARE SOLUTIONS

NOW IF OUR INITIAL COND. ARE:

$$y(0)=0, \quad y\left(\frac{\pi}{8}\right)=0$$

$$y(0)=c_1\cos 0 + c_2 \sin 0 = \underline{c_1 = 0}$$

$$y\left(\frac{\pi}{8}\right) = c_2 \underbrace{\sin \frac{\pi}{8}(4)}_{=1} = 0 \quad \text{so} \quad \underline{c_2=0}$$

AND OUR ONLY SOLUTION IS $y=0$

FINALLY IF:

$$y(0)=0, \quad y\left(\frac{\pi}{4}\right)=1$$

$$y(0)=c_1\cos 0 + c_2\sin 0 = \underline{c_1=0}$$

$$y\left(\frac{\pi}{4}\right) = c_2 \underbrace{\sin(\pi)}_{=0} = 1$$

$0=1$ CAN'T HAPPEN SO NO SOLUTION EXISTS

THE PROBLEM WAS OUR INITIAL CONDITIONS WERE NOT OF THE FORM $y(0)$, $y'(0)$ AS REQUIRED IN THE THM.

☆ THE SOLUTIONS OF A HOMOGENEOUS EQ. ARE A VECTOR SPACE!

SUPPOSE y_1 AND y_2 SATISFY $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$, (NO INITIAL CONDITIONS IMPOSED). THEN SO DO y_1+y_2 AND cy_1 FOR ANY $c\in\mathbb{R}$.

PF: $$a_n(y_1+y_2)^{(n)} + \ldots + a_1(y_1+y_2)' + a_0(y_1+y_2) = \underbrace{a_n y_1^{(n)} + \ldots + a_0 y_1}_{=0} + \underbrace{a_n y_2^{(n)} + \ldots + a_0 y_2}_{=0}$$

$$a_n(cy_1)^{(n)} + \ldots + a_0(cy_1) = c\left[\underbrace{a_n y^{(n)} + \ldots + a_0 y}_{=0}\right] \quad \blacksquare$$

SO WE CAN ADD AND SCALE ANY SOLUTIONS TO THE HOMOG. EQ AND GET ANOTHER.

THIS LEADS US NATURALLY TO:

DEF: A SET OF FUNCTIONS $f_1, \ldots, f_n$ ARE SAID TO BE LINEARLY INDEPENDENT IF WHEN $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$ (FOR ALL x) WE MUST HAVE $c_1 = c_2 = \ldots = c_n = 0$.

THIS IS THE SAME DEFINITION OF L.I. WE HAD BEFORE, BUT FOR FUNCTIONS.

EX: $f = \sin^2 x - 1$ AND $g = \frac{1}{2}\cos^2 x$ ARE NOT L.I. SINCE

$$f + 2g = \sin^2 x - 1 + \cos^2 x = 0 \text{ BY TRIG}$$

THERE IS A SIMPLE WAY TO TEST WHETHER OR NOT SEVERAL SOLUTIONS TO A HOMOG. $\underbrace{\text{DIFFY Q}}_{\text{WHAT I CALL DIFFERENTIAL EQUATIONS}}$ ARE L.I. – IT INVOLVES THE WRONSKIAN.

DEF: SUPPOSE $f_1, \ldots, f_n$ ARE FUNCTIONS OF X WITH AT LEAST $n-1$ DERIVATIVES EXISTING. THEN THE DETERMINANT:

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

IS CALLED THE WRONSKIAN OF THE FUNCTIONS.

THM: SUPPOSE $Y_1, \ldots, Y_n$ ARE SOLUTIONS TO A HOMOGENEOUS n-TH ORDER DIFFY Q ON SOME INTERVAL $I = (b, c)$. THEN THE SOLUTIONS $Y_1, \ldots, Y_n$ ARE L.I. IFF $W(Y_1, \ldots, Y_n) \neq 0$ FOR SOME POINT IN I.

PROOF: WE USE SOME LINEAR ALGEBRA AND IT'S NOT HARD – ASK ME IF YOU WANT TO KNOW.

BIG THM IF $Y_1 \ldots Y_n$ ARE L.I. SOLUTIONS TO A HOMOGENEOUS n-TH ORDER DIFFY Q, THEN THEY ARE A BASIS FOR THE VECTOR SPACE OF SOLUTIONS. IN OTHER WORDS, ANY SOLUTION Y WE LOOK LIKE:

$$Y = C_1 Y_1 + C_2 Y_2 + \cdots + C_n Y_n \quad \text{FOR SOME } C\text{'s}$$

PF: LET $Y(x)$ BE SOME SOLUTION TO:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

ON SOME INTERVAL $I = (b, c)$. THEN BY HYPOTHESIS, $Y_1 \ldots Y_n$ ARE L.I. SO THEIR WRONSKIAN $W(Y_1, \ldots, Y_n) \neq 0$ AT SOME $X_0 \in (b, c)$ (WRONSKIANS ARE FUNCTIONS NOT CONSTANTS).

THEN LOOK AT y AT x_0, AND DEFINE:

$$k_0 = y(x_0)\ ,\quad k_1 = y'(x_0),\quad k_2 = y''(x_0), \ \dots\ ,\quad k_{n-1} = y^{(n-1)}(x_0) \quad \left(\text{INITIAL CONDITIONS OF } y\right)$$

THE GOAL IS TO SHOW THAT FOR SOME $c_1, \dots, c_n$, THE FUNCTION

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

(WHICH IS A SOLUTION TO THE DIFFY Q) SATISFIES THE <u>SAME INITIAL CONDITIONS</u> AS y. THEN BY THE UNIQUENESS OF SOLUTIONS,

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n.$$

SINCE $W(y_1, \dots, y_n)(x_0) \neq 0$

WE HAVE
$$\begin{vmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \vdots & & & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{vmatrix} \neq 0$$

WHICH IMPLIES THAT THE SYSTEM OF EQ'S:

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \vdots & \vdots & & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} k_0 \\ \vdots \\ \\ k_{n-1} \end{pmatrix}$$

HAS A SOLUTION (IN FACT ONLY ONE SOLUTION) SINCE $\det A \neq 0 \iff A^{-1}$ EXISTS.

WHICH MEANS THAT THERE ARE SOME $c_1 \dots c_n$ s.t.

$$c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) = k_0 = y(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) = k_1 = y'(x_0)$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) = k_{n-1} = y^{(n-1)}(x_0)$$

IN OTHER WORDS, FOR THESE $c_1, \dots, c_n$ THE FUNCTION

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

SATISFIES THE <u>SAME INITIAL CONDITIONS AS y</u>. BY UNIQUESS OF SOLUTIONS $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

THIS THM IS VERY AWESOME. WHY? LET'S SAY WE WANTED TO SOLVE A 2ND ORDER HOMOGENEOUS DIFFY Q:

$$a_2(x)\, y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0$$

IF WE CAN USE SOME METHOD TO FIND TWO SOLUTIONS y_1 AND y_2 s.t. THEY ARE L.I., THEN ANY SOLUTION IS OF THE FORM $c_1 y_1 + c_2 y_2$.

EX: $y'' + 16y = 0$ SO $y'' = -16y$

LET'S SUPPOSE WE VERY SLICKLY NOTICE THAT BOTH $y_1 = \sin 4x$ AND $y_2 = \cos 4x$ SATISFY THE DIFFY Q. THEN WE CHECK THE WRONSKIAN:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 4x & \cos 4x \\ 4\cos 4x & -4\sin 4x \end{vmatrix} = -4\sin^2 4x - 4\cos^2 4x$$

$$= -4(\sin^2 4x + \cos^2 4x)$$

$$= -4 \neq 0 \text{ ANYWHERE}$$

THUS THEY ARE L.I. AND ANY SOLUTION IS OF THE FORM

$$y = c_1 \sin 4x + c_2 \cos 4x$$

NOW WHAT IF OUR DIFFY Q IS NOT HOMOGENEOUS? i.e. OF THE FORM

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = g$$

FOR SOME FUNCTION $g(x)$ WHICH IS NOT IDENTICALLY ZERO?

DEF: WE CALL A SOLUTION TO A NONHOMOGENEOUS SYSTEM (AS ABOVE) A PARTICULAR SOLUTION AND USUALLY DENOTE IT y_p.

THM: IF y_p AND $\tilde{y}_p$ ARE BOTH PARTICULAR SOLUTIONS OF THE NONHOMOG. DIFFY Q, THEN $y_p - \tilde{y}_p$ IS A SOLUTION TO:

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$$

NO g

THIS IS CALLED THE ASSOCIATED HOMOGENEOUS SYSTEM

PF: $$a_n(y_p - \tilde{y}_p)^{(n)} + \dots + a_0(y_p - \tilde{y}_p) = \Big[\underbrace{a_n y_p^{(n)} + \dots + a_0 y_p}_{=g}\Big] - \Big[\underbrace{a_n \tilde{y}_p^{(n)} + \dots + a_0 \tilde{y}_p}_{=g}\Big]$$

$$= 0 \quad \checkmark \quad \blacksquare$$

SO IF $y_1 \dots y_n$ ARE THE n L.I. SOLUTIONS TO THE HOMOG. SYSTEM, SINCE $y_p - \tilde{y}_p$ IS A SOLUTION TO THE HOMOG. SYSTEM THERE ARE SOME CONSTANTS $c_1, \dots, c_n$ s.t.

$$y_p - \tilde{y}_p = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

SO IN PARTICULAR, IF WE FIX $\tilde{y}_p$ AS SOME PARTICULAR SOLUTION THAT WE FIND (USING ANY METHOD WE WANT), THEN ANY OTHER PARTICULAR SOLUTION y_p LOOKS LIKE:

$$y_p = \tilde{y}_p + c_1 y_1 + \dots + c_n y_n$$

FOR THIS REASON, THIS IS CALLED THE GENERAL SOLUTION OF THE NONHOMOG. EQUATION.

EX: BACK TO OUR EXAMPLE, BUT WE CHANGE THE RIGHTHAND SIDE:

FIND THE GENERAL SOLUTION OF:

$$y'' + 16y = 32$$

JUST STARING AT IT FOR A WHILE, WE SEE THAT $y_p = 2$ IS A PARTICULAR SOLUTION SINCE:

$$y_p'' + 16 y_p = 0 + 16(2) = 32$$

WE ALREADY FOUND THE y_1 & y_2 SPANNING THE HOMOG. SOLUTIONS SO:

$$y = \underbrace{32}_{y_p} + \underbrace{c_1 \cos 4x + c_2 \sin 4x}_{\text{HOMOG.}}$$

IS THE GENERAL SOLUTION

SO THIS WILL BE OUR STRATEGY:

1. FIND THE HOMOG. SOLUTIONS $y_1, \dots, y_n$ THAT ARE L.I.
2. FIND y_p VIA SOME METHOD
3. $y = y_p + c_1 y_1 + \dots + c_n y_n$ ARE ALL SOLUTIONS.

HOMOGENEOUS LINEAR EQ'S WITH CONSTANT COEFFICIENTS

THESE ARE DIFFY Q'S OF THE FORM

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \quad \text{(HOMOG)}$$

WHERE THE a'S ARE ALL CONSTANTS.

FIRST LET'S TRY $n=2$, I.E. SOLVING

$$ay'' + by' + cy = 0$$

WE GUESS THAT A SOLUTION OF THE FORM $y = e^{mx}$ WORKS (A REASONABLE GUESS SINCE ITS DERIVATIVES ARE ALL JUST MULTIPLES OF y)

PLUG-IN:

$$\left.\begin{aligned} y &= e^{mx} \\ y' &= m e^{mx} \\ y'' &= m^2 e^{mx} \end{aligned}\right\} \quad \begin{aligned} &a m^2 e^{mx} + b m e^{mx} + a e^{mx} = 0 \\ &(am^2 + bm + c)\underbrace{e^{mx}}_{\neq 0} = 0 \end{aligned}$$

SO $am^2 + bm + c = 0$, SO

$$m = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

THERE ARE 3 CASES:

CASE 1 DISTINCT REAL ROOTS ($b^2 - 4ac > 0$)

IN THIS CASE m HAS TWO SOLUTIONS m_1 AND m_2 (NOT EQUAL)

SO LET $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$

THESE ARE L.I. SOLUTIONS: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$

$$= (m_2 - m_1) e^{(m_1 + m_2)x} \neq 0$$

(SINCE $m_2 \neq m_1$)

THUS ANY SOLUTION IS OF THE FORM:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

CASE 2 REPEATED ROOTS ($b^2 - 4ac = 0$)

IN THIS CASE $m_1 = m_2$. OUR ONLY SOLUTION WE FIND IS $y = e^{m_1 x}$.

<u>BUT</u>, WE CAN LET $y_2 = x e^{m_1 x}$, IT STILL SOLVES THE EQUATION. (CHECK!)

ALSO, THESE ARE L.I. SOLUTIONS:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & x e^{m_1 x} \\ m_1 e^{m_1 x} & e^{m_1 x} + m_1 x e^{m_1 x} \end{vmatrix}$$

$$= e^{2m_1 x} + \cancel{m_1 x e^{2m_1 x}} - \cancel{m_1 x e^{2m_1 x}}$$

$$= e^{2m_1 x} \neq 0$$

SO ANY SOLUTION IS OF THE FORM:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

CASE 3 COMPLEX ROOTS $(b^2 - 4ac < 0)$

HERE $m_1 = \alpha + i\beta$ AND $m_2 = \alpha - i\beta$ (ROOTS COME IN CONJUGATE PAIRS)

FOR THE SAME REASONS $y_1 = e^{m_1 x}$ AND $y_2 = e^{m_2 x}$ ARE L.I. SO

ANY SOLUTION IS OF THE FORM:

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

BUT, WE TYPICALLY LIKE TO GET RID OF THE i'S IN OUR SOLUTIONS SO THAT WE CAN PICTURE THE SOLUTIONS (THESE SOLUTIONS ARE IN FACT SINUSOIDAL!)

WE DO THIS BY:

EULER'S FORMULA FOR $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i \sin\theta$

HOW DO WE EVEN MAKE SENSE OF e^{ix}? WE USE THE POWER SERIES FOR e^x.

THIS IS WHERE EULER'S FORMULA COMES FROM.

RECALL: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n \geq 0} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots = \sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots = \sum_{n \geq 0} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \dots \quad (\text{SINCE } i^2 = -1)$$

NOW GROUP REAL & IMAGINARY:

$$= \Big[\underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos\theta}\Big] + i\Big[\underbrace{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots}_{\sin\theta}\Big]$$

$$= \cos\theta + i\sin\theta$$

SO $Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$

$Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos\beta x + i\sin(-\beta x))$

$= e^{\alpha x}(\cos\beta x - i\sin\beta x)$

RECALL: $\sin(-\theta) = -\sin\theta$

ANY LINEAR COMBINATION OF Y_1 AND Y_2 ARE SOLUTIONS, SO CONSIDER:

$$\frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}(2e^{\alpha x}\cos\beta x) = e^{\alpha x}\cos\beta x$$

$$\frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}(2i\, e^{\alpha x}\sin\beta x) = e^{\alpha x}\sin\beta x$$

SO $W_1 = e^{\alpha x}\cos\beta x$, $W_2 = e^{\alpha x}\sin\beta x$ ARE ALSO SOLUTIONS,

AND $W(W_1, W_2) \neq 0$ (CHECK!) SO THEY ARE L.I.

THUS ANY SOLUTION IS OF THE FORM:

$$Y = c_1 e^{\alpha x}\sin\beta x + c_2 e^{\alpha x}\cos\beta x$$

WHERE $m_1 = \alpha + i\beta$
$m_2 = \alpha - i\beta$

THIS SOLVES ANY 2ND ORDER DIFFY Q WITH CONSTANT COEFFS!

EX: FIND GENERAL SOLUTION: $2y'' - 5y' - 3y = 0$

GUESS $y = e^{mx}$, GET: $2m^2 - 5m - 3 = 0$

$$2m^2 - 6m + m - 3 = 0$$

$$2m(m-3) + (m-3) = 0$$

$$(2m+1)(m-3) = 0$$

$$m = 3 \text{ OR } m = -\tfrac{1}{2} \quad \text{CASE 1}$$

$$y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}$$

EX: $y'' - 10y' + 25y = 0$

GUESS $y = e^{mx}$: $m^2 - 10m + 25 = 0$

$$(m-5)^2 = 0$$

$$m = 5, 5 \quad \text{CASE 2}$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

EX: $y'' + 4y' + 7y = 0$

GUESS $y = e^{mx}$: $m^2 + 4m + 7 = 0$

$$m = \tfrac{1}{2}\left(-4 \pm \sqrt{16 - 4(7)}\right)$$

$$= \tfrac{1}{2}\left(-4 \pm \sqrt{-12}\right)$$

$$= \tfrac{1}{2}\left(-4 \pm 2\sqrt{3}\, i\right)$$

$$= \underset{\alpha}{-2} \pm \underset{\beta}{\sqrt{3}}\, i \quad \text{CASE 3}$$

$$y = c_1 e^{-2x} \cos\sqrt{3}x + c_2 e^{-2x} \sin\sqrt{3}x$$

NOW IF SOME INITAL VALUES ARE SPECIFIED, WE CAN PICK OUT VALUES FOR c_1 & c_2 TO GET A SOLUTION:

SOLVE: $2y'' - 5y' - 3y = 0$ $\quad y(0) = 0 \quad y'(0) = 1$

WE GOT THE GENERAL SOLUTION ABOVE: $y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}$, $y' = 3c_1 e^{3x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x}$

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \quad \text{SO} \quad \underline{c_1 = -c_2}$$

$$y'(0) = 3c_1 e^0 - \tfrac{1}{2} c_2 e^0 = 3c_1 - \tfrac{1}{2} c_2 = 1 \quad \text{SO} \quad \tfrac{7}{2} c_1 = 1, \quad c_1 = \tfrac{2}{7} \quad c_2 = -\tfrac{2}{7}$$

AND $y = \frac{2}{7} e^{3x} - \frac{2}{7} e^{-\frac{1}{2}x}$

FOR HIGHER ORDER EQUATIONS WITH CONSTANT COEFFICIENTS, WE DO THE SAME THING, i.e. GUESS $y=e^{mx}$ FOR SOMETHING OF THE FORM:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

IF OUR POLYNOMIAL IS SOMETHING LIKE:

$$(m-1)^5 (m-2)^2 = 0 \quad , \quad m = 1,1,1,1,1,2,2$$

THEN FOR EACH REPEATED ROOT WE MULTIPLY OUR SOL'S BY x, FOR EXAMPLE IN THIS CASE A GENERAL SOL. WOULD BE:

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x + c_5 x^4 e^x + c_6 e^{2x} + c_7 x e^{2x}$$

IF WE GET SOMETHING LIKE:

$$(m^2+1)^2 (m-3) = 0, \quad m = i, i, -i, -i, 3$$

THEN OUR SOLUTION WOULD BE:

$$y = c_1 e^{ix} + c_2 e^{-ix} + c_3 x e^{ix} + c_4 x e^{-ix} + c_5 e^{3x}$$

$$= c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + c_5 e^{3x} \quad (\text{FOR SOLUTIONS IN } \mathbb{R})$$

EX: FIND THE GENERAL SOLUTION: $y''' - 3y' + 2y = 0$

GUESS $y = e^{mx}$: $\quad (m^3 - 3m + 2)\underbrace{e^{mx}}_{\neq 0} = 0$

$$m^3 - 3m + 2 = 0$$

ONLY POSSIBLE RATIONAL ROOTS ARE $\dfrac{\text{FACTORS OF CONSTANT COEFFICIENT}}{\text{FACTORS OF LEADING } (m^3) \text{ COEFF.}} = \dfrac{\pm 1, \pm 2}{\pm 1}$

$= \{\pm 1, \pm 2\}$ ONLY POSSIBLE RATIONAL ROOTS

THIS IS THE RATIONAL ROOT TEST (WIKIPEDIA IT)

SO WE TEST THESE, FIND THAT $m=-2$ IS A ROOT,

SO $m^3 - 3m + 2 = (m+2)(\ ?\ )$

TO FIND ?, DO POLYNOMIAL LONG DIVISION:

$$\begin{array}{r|l} & m^2 - 2m + 1 \\ \hline m+2 & m^3 + 0m^2 - 3m + 2 \\ & \underline{m^3 + 2m^2} \\ & -2m^2 - 3m \\ & -(\underline{-2m^2 - 4m}) \\ & m + 2 \end{array}$$

SO $m^3 - 3m + 2 = (m+2)(m^2 - 2m + 1)$

$= (m+2)(m-1)^2$

$m = -2, 1, 1$

$$\boxed{y = c_1 e^{-2x} + c_2 e^x + c_3 x e^x}$$

THIS IDEA OF THE GENERAL SOLUTION TAKING THE FORM $Y = Y_P + Y_H$ OCCURS IN LINEAR ALGEBRA AS WELL, AND CAN BE USEFUL IN CHECKING YOUR SOLUTIONS.

SUPPOSE WE ARE SOLVING $Ax = b$ (SOME SYSTEM OF EQ'S).

WE CALL X_p A PARTICULAR SOLUTION IF $AX_p = b$, AND A HOMOGENEOUS SOLUTION X_H IS WHEN $AX_H = 0$. AGAIN, IF X_p AND $\tilde{X}_p$ ARE PARTICULAR SOLUTIONS THEN THEIR DIFFERENCE $X_p - \tilde{X}_p$ IS A HOMOGENEOUS SOL.:

$$A(X_p - \tilde{X}_p) = AX_p - A\tilde{X}_p = b - b = 0$$

ALSO AGAIN, THE SOLUTION TO $Ax = 0$ IS A VECTOR SPACE (WHOSE DIMENSION IS THE # OF PARAMETERS) SO WE CAN FIND L.I. VECTORS $X_1, \ldots, X_r$ (r = # PARAMS) s.t. $AX_1 = 0$, $AX_2 = 0$, ...

THEN THE GENERAL SOLUTION TO $Ax = b$ IS:

$$X = X_p + \underbrace{c_1X_1 + c_2X_2 + \ldots + c_rX_r}_{\text{HOMOG. SOL}}$$

EX: ON PG 27 WE SOLVED A SYSTEM OF EQ'S:

$$\underbrace{\begin{pmatrix} -1 & 2 & 2 & 0 \\ 1 & -2 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} 4 \\ -4 \\ 1 \\ -3 \end{pmatrix}}_{b} \quad \text{SO} \quad Av = b$$

AND FOUND THE SOLUTION TO BE:

$$\begin{pmatrix} -2 - 4z - 2w \\ 1 - 3z - w \\ z \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{X_p} + \underbrace{z\begin{pmatrix} -4 \\ -3 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{X_H}$$

X_p IS THE CONSTANT PART

X_H IS THE PART WITH PARAMETERS

CHECK THAT $AX_p = b$

AND $AX_H = 0$ (IN FACT $A\begin{pmatrix} -4 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$)

THIS GIVES US AN EASY WAY TO CHECK OUR SOLUTIONS, JUST SPLIT THE SOLUTION UP AS ABOVE AND CHECK THE TWO ABOVE FORMULAS.