LAST TIME: WE TALKED ABOUT SOLVING n-TH ORDER LINEAR DIFFERENTIAL EQ'S:

$$\bigstar \; a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g$$

HERE $g, a_0, a_1, \dots, a_n$ ARE FUNCTIONS OF x. WHAT WE DETERMINED IS THAT THE GENERAL SOLUTION WAS OF THE FORM:

$$y = y_p + c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

WHERE y_p SATISFIES $\bigstar$. AND $y_1 \dots y_n$ SATISFY

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \quad (\text{SAME AS } \bigstar \text{ BUT } g=0)$$

(ASSOCIATED HOMOG. EQUATION)

WE ALSO SHOWED HOW TO DETECT IF WE FOUND L.I. SOLUTIONS $y_1 \dots y_n$ (WRONSKIAN)

WE THEN FOCUSED ON THE CASE OF 2ND ORDER EQ'S WITH a_0, a_1, AND a_2 BEING CONSTANT FUNCTIONS AND $g=0$:

$$ay'' + by' + cy = 0$$

AND FOUND THAT SOLUTIONS OF THE FORM $y = e^{mx}$ WORK. WE SOLVE FOR m VIA

$$am^2 + bm + c = 0 \quad \text{HAS ROOTS } m_1 \,\&\, m_2$$

$$m = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

WE HAD 3 CASES:

① $m_1 \neq m_2$ REAL ROOTS
$(b^2 - 4ac > 0)$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{GEN. SOL.}$$

② $m_1 = m_2$
$(b^2 - 4ac = 0)$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

③ $m_1 \neq m_2$ COMPLEX ROOTS
$(b^2 - 4ac < 0)$

$$m_1 = \alpha + i\beta$$
$$m_2 = \alpha - i\beta$$

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

<u>NOW</u> WE WILL TALK ABOUT SOLVING SOMETHING NONHOMOGENEOUS:

$$ay'' + by' + cy = g(x)$$

THEN GENERAL SOLUTION IS OF THE FORM:

$$y = \underset{\substack{\uparrow \\ \text{HOW DO WE FIND THIS?}}}{y_p} + \underbrace{c_1 y_1 + c_2 y_2}_{\text{WE CAN FIND THESE SINCE THEY ARE THE SOLUTIONS OF } ay'' + by' + cy = 0 \text{ USING ABOVE}}$$

NOW THAT WE CAN FIND HOMOGENEOUS SOLUTIONS, THE NEXT STEP IS TO BE ABLE TO FIND THE PARTICULAR SOLUTION Y_P OF A NONHOMOG. DIFFY Q. THIS BRINGS US TO:

UNDETERMINED COEFFICIENTS

THIS IS A METHOD FOR SOLVING NONHOMOGENEOUS DIFFY Q'S W/ CONSTANT COEFFICIENTS. WE JUST GUESS WHAT COULD WORK!

EX: FIND A PARTICULAR SOLUTION OF:

$$y'' - 2y' + y = 3x^2 - 11x + 4$$

A REASONABLE GUESS FOR Y WOULD BE A POLYNOMIAL (BASED ON THE RIGHT SIDE OF THE EQ).

LET'S GUESS: $Y = Ax^2 + Bx + C$ FOR SOME A, B AND C

THEN $Y' = 2Ax + B$

$Y'' = 2A$

AND WE PLUG IN:

$$y'' - 2y' + y = \underbrace{2A - 2(2Ax + B) + (Ax^2 + Bx + C)}_{\text{GROUP LIKE TERMS \& EQUATE TO}} = 3x^2 - 11x + 4$$

$$\underbrace{Ax^2}_{=3} + \underbrace{(B - 4A)}_{=-11}x + \underbrace{(2A - 2B + C)}_{=4} = 3x^2 - 11x + 4$$

SO $A = 3$, $B - 4A = -11$ $\quad$ $2A - 2B + C = 4$

$B = 1$ $\quad$ $6 - 2 + C = 4$

$C = 0$

SO $\boxed{Y_P = 3x^2 + x}$ WORKS (CHECK!)

WHILE WE'RE AT IT, WE CAN FIND THE GENERAL SOLUTION IF WE FIND THE TWO SOLUTIONS TO THE HOMOGENEOUS SYSTEM:

$$Y'' - 2Y' + Y = 0$$

$Y = e^{mx}$: $\quad m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

$Y_H = c_1 e^x + c_2 x e^x$ HOMOG. SOLUTION

$Y = Y_P + Y_H$

$\boxed{Y = 3x^2 + x + c_1 e^x + c_2 x e^x}$

GENERAL SOLUTION

SO, HOW DO WE KNOW WHAT TO GUESS? IN GENERAL, FOR GUESSING Y_P TO THE EQ:

$$ay'' + by' + cy = g(x)$$

LOOK AT EACH TERM IN g, AND CONSIDER ALL OF ITS DERIVATIVES $g', g'', g''', \ldots$ ETC. AND GIVE Y_P A TERM CORRESPONDING TO ANY OF THE TERMS YOU SEE IN THESE DERIVATIVES.

EX: IF $g = \cos x$ GUESS $Y_P = A\cos x + B\sin x$

$g', g'', g''' \ldots$ ARE ALL $\sin x$ & $\cos x$

IF $g = x^3 - x$ GUESS $Y_P = Ax^3 + Bx^2 + Cx + D$

IF $g = xe^x$ GUESS $Y_P = Axe^x + Be^x$

$g' = e^x + xe^x$

$g'' = 2e^x + xe^x \ldots$ (ONLY e^x AND xe^x TERMS)

IF $g = \cos x + x^2$ GUESS $Y_P = Ax^2 + Bx + C + D\cos x + E\sin x$

IF $g = e^x \cos x$ GUESS $Y_P = Ae^x\cos x + Be^x \sin x$

$g' = e^x \cos x - e^x \sin x$

$\vdots$

ONLY $e^x\cos x$, $e^x \sin x$ TERMS

EX: FIND A PARTICULAR SOLUTION TO $y'' - 4y = 2\cos x + x$

GUESS $Y_P = A\cos x + B\sin x + Cx + D$

$Y_P' = -A\sin x + B\cos x + C$

$Y_P'' = -A\cos x - B\sin x$

PLUG IN: $Y_P'' - 4Y_P = (-A\cos x - B\sin x) - 4(A\cos x + B\sin x + Cx + D) = 2\cos x + x$

$$-5A\cos x - 5B\sin x - 4Cx - 4D = 2\cos x + x$$

EQUATE LIKE TERMS: $-5A = 2$ $\quad -5B = 0 \quad -4C = 1 \quad -4D = 0$

$A = -\frac{2}{5} \quad B = 0 \quad C = -\frac{1}{4} \quad D = 0$

SO $Y_P = -\frac{2}{5}\cos x - \frac{1}{4}x$ IS A SOLUTION

NOW LET'S FIND THE GENERAL SOLUTION BY SOLVING THE ASSOCIATED HOMOG. EQ:

$y'' - 4y = 0$ LET $y = e^{mx}$, GET $m^2 - 4 = 0$

$m = \pm 2$

SO $Y_H = C_1 e^{2x} + C_2 e^{-2x}$

SO THE GENERAL SOLUTION IS:

$$y = y_P + y_H$$

$$y = -\frac{2}{5}\cos x - \frac{1}{4}x + c_1 e^{2x} + c_2 e^{-2x}$$

AGAIN THIS IS AN ENTIRE FAMILY OF SOLUTIONS SO WE CAN IMPOSE SOME INITIAL CONDITIONS TO GET VALUES FOR c_1 & c_2

THERE IS ONE OTHER THING WE NEED TO BE CAREFUL OF WHEN CHOOSING y_P. HERE IS AN EXAMPLE:

EX: FIND THE GENERAL SOLUTION OF:

$$y'' + y' - 2y = 3e^x$$

FIRST LET'S FIND y_H:

$$m^2 + m - 2 = 0$$

$$(m+2)(m-1) = 0$$

$$m = -2, 1$$

SO $y_H = c_1 e^x + c_2 e^{-2x}$

NOW TO GUESS y_P, ONE WOULD THINK $y_P = Ae^x$ WOULD BE FINE.

THEN $y_P' = y_P'' = y_P = Ae^x$ AND WE PLUG IN:

$$y_P'' + y_P' - 2y_P = Ae^x + Ae^x - 2Ae^x = 3e^x$$

$$0 = 3e^x$$

SO WHAT HAPPENED? WE PLUGGED IN y_P INTO THE DIFFYQ AND GOT ZERO! THIS IS THE DEFINITION OF A HOMOGENEOUS SOLUTION, OF WHICH WE KNOW <u>ALL</u> OF THEM: $y_H = c_1 e^x + c_2 e^{-2x}$ ARE ALL OF THEM.

SO, <u>THE PROBLEM WAS THAT OUR GUESS FOR y_P WAS IN y_H</u>.

WHAT DO WE DO NOW?

RULE: WHEN ALL ELSE FAILS, JUST MULTIPLY BY X.

LET'S GUESS $y_P = Axe^x$

$$y_P' = A[e^x + xe^x]$$

$$y_P'' = A[2e^x + xe^x]$$

PLUG THIS IN:

$$y_p'' + y_p' - 2y_p = A[2e^x + xe^x] + A[e^x + xe^x] - 2Axe^x = 3e^x$$

GROUP LIKE TERMS

$$[2A + A]e^x + [A + A - 2A]xe^x = 3e^x$$

$$3Ae^x = 3e^x$$

$$\underline{\text{SO } A = 1} \qquad y_p = xe^x$$

AND OUR GENERAL SOLUTION IS:

$$y = y_p + y_H$$

$$y = xe^x + c_1e^x + c_2e^{-2x}$$

EX: FIND THE GENERAL SOLUTION OF:

$$y'' - 2y' + y = e^x$$

FIRST FIND y_H:

$$m^2 - 2m + 1 = 0 \qquad y_H = c_1e^x + c_2xe^x$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

NOW FOR y_p WE INITIALLY WANT TO GUESS $\underline{y_p = Ae^x}$ BUT <u>DON'T</u> SINCE IT IS IN y_H. WE ALSO CAN'T GUESS $\underline{y_p = Axe^x}$ SINCE IT IS ALSO IN y_H.

SO...

RULE: STILL IN DOUBT? MULTIPLY BY X SOME MORE!

GUESS $y_p = Ax^2e^x$

$$y_p' = A[2xe^x + x^2e^x] = A[2x + x^2]e^x$$

$$y_p'' = A[(2+2x)e^x + (2x + x^2)e^x] = A[2 + 4x + x^2]e^x$$

PLUG-IN: $y_p'' - 2y_p' + y_p = e^x$

$$A[2+4x+x^2]e^x - 2A[2x + x^2]e^x + Ax^2e^x = e^x$$

$$2Ae^x = e^x$$

$$A = \frac{1}{2} \qquad y_p = \frac{1}{2}x^2e^x$$

$$y = y_p + y_H$$

$$y = \frac{1}{2}x^2e^x + c_1e^x + c_2xe^x \quad \text{GEN. SOL.}$$

IT IS IMPORTANT TO NOT BE OVERZEALOUS IN YOUR MULTIPLYING-BY-X-NESS.

IF WE HAD GUESSED $Y = Ax^3 e^x$ IT WOULD NOT HAVE WORKED.

ONLY MULTIPLY BY X UNTIL Y_P IS NO LONGER IN Y_H.

FOR THIS REASON YOU ALWAYS WANT TO FIND Y_H FIRST.

WHEN SOLVING $ay'' + by' + cy = g$

IF g IS COMPLICATED WE CAN BREAK IT UP INTO PIECES AND FIND A Y_P FOR EACH "PIECE":

EX: FIND THE GENERAL SOLUTION OF:

$$y'' + 2y' - 3y = \overbrace{x^3 + 2x^2}^{g_1} + \overbrace{2\cos x + \sin x}^{g_2} \overbrace{- e^x}^{g_3}$$

Y_H: $m^2 + 2m - 3 = 0$

$(m-1)(m+3) = 0$

$m = 1, -3$

$$Y_H = c_1 e^x + c_2 e^{-3x}$$

WE SPLIT UP g INTO $g_1 + g_2 + g_3$

NOW WE FIND A Y_{P_1} s.t. $Y_{P_1}'' + 2Y_{P_1}' - 3Y_{P_1} = x^3 + 2x^2 = g_1$

AND FIND A Y_{P_2} s.t. $Y_{P_2}'' + 2Y_{P_2}' - 3Y_{P_2} = 2\cos x + \sin x = g_2$

" Y_{P_3} s.t. $Y_{P_3}'' + 2Y_{P_3}' - 3Y_{P_3} = -e^x = g_3$

ADDING THESE THREE EQ'S:

$$(Y_{P_1} + Y_{P_2} + Y_{P_3})'' + 2(Y_{P_1} + Y_{P_2} + Y_{P_3})' - 3(Y_{P_1} + Y_{P_2} + Y_{P_3}) = g_1 + g_2 + g_3 = g$$

☆ i.e. $Y_P = Y_{P_1} + Y_{P_2} + Y_{P_3}$ IS OUR FINAL PARTICULAR SOLUTION

GUESS: $Y_{P_1} = Ax^3 + Bx^2 + Cx + D$

$Y_{P_1}' = 3Ax^2 + 2Bx + C$

$Y_{P_1}'' = 6Ax + 2B$

PLUG-IN: $(6Ax + 2B) + 2(3Ax^2 + 2Bx + C) - 3(Ax^3 + Bx^2 + Cx + D) = x^3 + 2x^2$

$$-3Ax^3 + (6A - 3B)x^2 + (6A + 4B - 3C)x + (2B + 2C - 3D)$$

$$\| $$

$$x^3 + 2x^2$$

SO $-3A = 1$ | $6A - 3B = 2$ | $6A + 4B - 3C = 0$ | $2B + 2C - 3D = 0$

$A = -\frac{1}{3}$ | $-2 - 3B = 2$ | $-2 - \frac{16}{3} = 3C$ | $-\frac{8}{3} - \frac{44}{9} = 3D$

$B = -\frac{4}{3}$ | $-\frac{22}{9} = C$ | $-\frac{68}{27} = D$

So $y_{p_1} = -\frac{1}{3}x^3 - \frac{4}{3}x^2 - \frac{22}{9}x - \frac{68}{27}$

GUESS $y_{p_2} = A\cos x + B\sin x$

$y_{p_2}' = -A\sin x + B\cos x$

$y_{p_2}'' = -A\cos x - B\sin x$

PLUG-IN: $(-A\cos x - B\sin x) - 2(-A\sin x + B\cos x) - 3(A\cos x + B\sin x) = 2\cos x + \sin x$

$(-4A - 2B)\cos x + (2A - 4B)\sin x = 2\cos x + \sin x$

So $\begin{matrix} -4A - 2B = 2 \\ 2A - 4B = 1 \end{matrix} \xrightarrow{\text{DOUBLE } E_2} \left.\begin{matrix} -4A - 2B = 2 \\ 4A - 8B = 2 \end{matrix}\right\}$ ADD $-10B = 4$

$B = -\frac{2}{5}$

$2A = 1 + 4B$

$A = \frac{1}{2}\left(1 - \frac{8}{5}\right)$

$A = -\frac{3}{10}$

So $y_{p_2} = -\frac{3}{10}\cos x - \frac{2}{5}\sin x$

FOR y_{p_3}, GUESS $y_{p_3} = Axe^x$ SINCE e^x IS IN y_H

$y_{p_3}' = A[e^x + xe^x] = Ae^x(x+1)$

$y_{p_3}'' = Ae^x(x+1) + Ae^x = Ae^x(x+2)$

PLUG-IN: $Ae^x(x+2) + 2Ae^x(x+1) - 3Axe^x = -e^x$

$4Ae^x = -e^x$

$A = -\frac{1}{4}$

$y_{p_3} = -\frac{1}{4}xe^x$

So OUR FINAL AWFUL SOLUTION IS:

$$y = \underbrace{-\frac{1}{3}x^3 - \frac{4}{3}x^2 - \frac{22}{9}x - \frac{68}{27}}_{y_{p_1}} \underbrace{- \frac{3}{10}\cos x - \frac{2}{5}\sin x}_{y_{p_2}} \underbrace{- \frac{1}{4}xe^x}_{y_{p_3}} + \underbrace{c_1e^x + c_2e^{-3x}}_{y_H}$$

IS THE GENERAL SOLUTION. WITHOUT SPLITTING UP y_p THIS WOULD HAVE BEEN A NIGHTMARE.

SO LETS MAKE CLEAR EXACTLY WHAT TO GUESS WHEN WHAT WE WANT TO GUESS FOR Y_P IS IN Y_H. IN OUR EARLIER EXAMPLE WE HAD:

$$y'' - 2y' + y = e^{x}$$
$$Y_H = c_1 e^{x} + c_2 x e^{x}$$

WHAT WE WOULD NAIVELY GUESS FOR Y_P IS Ae^{x}, BUT WE NEEDED TO MULTIPLY THIS BY x^2 IN ORDER TO NOT HAVE PART OF Y_P BEING IN Y_H. NOW HERE IS ANOTHER EXAMPLE:

EX: $y^{(4)} + y''' = -x^2 e^{-x}$ FIND THE GENERAL SOLUTION.

Y_H:
$$m^4 + m^3 = 0$$
$$m^3(m+1) = 0$$
$$m = 0, 0, 0, -1$$

$$Y_H = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + c_4 e^{-x}$$
$$Y_H = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

OUR NAIVE GUESS WOULD BE: $Y_P = Ax^2 e^{-x} + Bx e^{-x} + Ce^{-x}$

<u>BUT</u> THIS DUPLICATES THE e^{-x} TERM IN Y_H. SO, WE MULTIPLY <u>OUR WHOLE Y_P</u> BY x AND GUESS: $Y_P = Ax^3 e^{-x} + Bx^2 e^{-x} + Cx e^{-x}$

RATHER THAN JUST DROPPING OUT THE Ce^{-x} TERM AND GUESSING:

$$Y_P = Ax^2 e^{-x} + Bx e^{-x}$$

WE WILL NEED THE $Ax^3 e^{-x}$ TERM! HERE IS ANOTHER EXAMPLE:

EX: $y'' + y = x^2 \cos x$

Y_H:
$$m^2 + 1 = 0$$
$$m^2 = -1$$
$$m = \pm i$$

$$Y_H = C_1 \cos x + C_2 \sin x$$

REPEATS OF Y_H

OUR NAIVE GUESS FOR Y_P IS $Y_P = Ax^2\cos x + Bx^2 \sin x + Cx\cos x + Dx\sin x + E\cos x + F\sin x$

THUS WE MULTIPLY ALL OF Y_P BY x AND CHANGE OUR GUESS TO:

$$Y_P = Ax^3\cos x + Bx^3 \sin x + Cx^2\cos x + Dx^2\sin x + Ex\cos x + Fx\sin x$$

THIS MIGHT NOT EVEN COME UP BUT IT IS GOOD TO KNOW.

CAUCHY-EULER EQUATION

HERE WE WILL TALK ABOUT A CERTAIN CASE WHEN WE DON'T HAVE CONSTANT COEFFICIENTS IN OUR DIFFY Q'S BUT CAN STILL SOLVE THEM.

EX: FIND THE GENERAL SOLUTION TO:

$$x^2y'' - 4xy' + 6y = 0$$

HERE LET'S GUESS $y = x^m$ FOR SOME m. THEN:

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

PLUG THESE IN:

$$x^2\left(m(m-1)x^{m-2}\right) - 4x\left(mx^{m-1}\right) + 6x^m = 0$$

$$\left[m(m-1) - 4m + 6\right] x^m = 0$$

($= 0$ FOR ALL x SO WE WANT THE POLYNOMIAL IN m TO EQUAL ZERO)

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

SO $y_1 = x^2$, $y_2 = x^3$ SOLVE THE DIFFY Q

ARE THEY L.I.?

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 = 0 \text{ ONLY AT } x=0$$

SO THESE SOLUTIONS ARE L.I. ANYWHERE OUTSIDE OF $x = 0$.

SO ANY SOLUTION IS OF THE FORM:

$$y = c_1x^2 + c_2x^3$$

SO, WHEN DOES THIS WORK? THE KEY IS THAT WE COULD FACTOR OUT ALL OF THE X'S. WE DID THIS SINCE EACH TERM HAD X TO THE SAME POWER. EACH DERIVATIVE DROPPED THE POWER OF x^m BY ONE, AND THEN EACH X IN THE COEFFICIENT RAISED THE POWER BY 1.

$$\underset{\substack{\uparrow \\ \text{UP} \\ 2}}{x^2}\ \underset{\substack{\uparrow \\ \text{DOWN} \\ 2}}{y''} - 4\underset{\substack{\uparrow \\ \text{UP} \\ 1}}{x}\ \underset{\substack{\uparrow \\ \text{DOWN} \\ 1}}{y'} + 6y = 0$$

SO THE NET # OF X'S WAS THE SAME ON EACH TERM.

EX: WHICH CAN YOU SOLVE USING THE SUBSTITUTION $y = x^m$?

① $x^3 y'' + 3xy = 0$

② $2xy'' + x^2 y' = 0$

③ $xy'' + 4y' = 0$

ANS: ① AND ③

IN ① $\underbrace{\underset{\text{UP }3}{x^3}\ \underset{\text{DOWN }2}{y''}}_{\text{UP }1} + \underbrace{3xy}_{\text{UP }1}$; $\text{UP }1 = \text{UP }1$

② $\underbrace{\underset{\text{UP }1}{2x}\ \underset{\text{DOWN }2}{y''}}_{\text{DOWN }1} + \underbrace{\underset{\text{UP }2}{x^2}\ \underset{\text{DOWN }1}{y'}}_{\text{UP }1}$; $\text{DOWN }1 \neq \text{UP }1$

③ $\underbrace{\underset{\text{UP }1}{x}\ \underset{\text{DOWN }2}{y''}}_{\text{DOWN }1} + \underbrace{4y'}_{\text{DOWN }1}$; $\text{DOWN }1 = \text{DOWN }1$

OF COURSE IF IN DOUBT JUST TRY $y = x^m$ AND SEE IF IT FACTORS OUT.

SO AGAIN JUST THINKING ABOUT 2ND-ORDER DIFFY Q'S, IF WE GUESS $y = x^m$ WE WILL GET SOME 2ND DEGREE POLYNOMIAL IN m (2 ROOTS). AGAIN, WE WILL HAVE 3 CASES:

CASE 1 $m_1 \neq m_2$ THEN $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$ AS BEFORE

CASE 2 $m_1 = m_2$ THEN $\boxed{y = c_1 x^{m_1} + c_2 (\ln x) x^{m_1}}$

SO $y_1 = x^{m_1}$. TO FIND $y_2 = (\ln x) x^{m_1}$, WE ASSUME IT LOOKS LIKE $u(x)\, y_1(x)$ FOR SOME FUNCTION $u(x)$ AND FIND $u(x) = \ln x$ BY <u>REDUCTION OF ORDER</u> (SECT. 3.2) THIS IS NOT ON OUR SYLLABUS!

CASE 3 $m_1 = \alpha + i\beta$
$m_2 = \alpha - i\beta$
COMPLEX, DISTINCT ROOTS

THEN $\boxed{y = c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta}}$ IS A GENERAL SOLUTION BUT AGAIN WE WANT <u>REAL</u> SOLUTIONS SO WE APPLY EULER'S FORMULA AGAIN.

SINCE $X = e^{\ln x}$ FOR $x > 0$,

$$Y_1 = x^{\alpha + i\beta} = x^{\alpha} x^{i\beta} = x^{\alpha}\left(e^{\ln x}\right)^{i\beta} = x^{\alpha} e^{i\beta \ln x} \xrightarrow{\text{EULER}} = x^{\alpha}\left(\cos(\beta \ln x) + i \sin(\beta \ln x)\right)$$

$$Y_2 = x^{\alpha - i\beta} = x^{\alpha} x^{-i\beta} = x^{\alpha}\left(e^{\ln x}\right)^{-i\beta} = x^{\alpha} e^{-i\beta \ln x} = x^{\alpha}\left(\cos(-\beta \ln x) + i \sin(-\beta \ln x)\right)$$

$$= x^{\alpha}\left(\cos(\beta \ln x) - i \sin(\beta \ln x)\right)$$

ANY LINEAR COMBINATION OF Y_1 & Y_2 IS A SOLUTION...

SO THEN:

$$W_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2x^{\alpha}\cos(\beta \ln x)\right) = x^{\alpha}\cos(\beta \ln x)$$

$$W_2 = \frac{1}{2i}(Y_1 + Y_2) = \frac{1}{2i}\left(2i\, x^{\alpha}\sin(\beta \ln x)\right) = x^{\alpha}\sin(\beta \ln x)$$

W_1 AND W_2 ARE BOTH <u>REAL</u> SOLUTIONS AND ARE L.I. SINCE $W(w_1, w_2) \neq 0$ FOR $x > 0$

<u>THUS</u> WE CAN ALSO WRITE

$$Y = c_1 x^{\alpha}\cos(\beta \ln x) + c_2 x^{\alpha}\sin(\beta \ln x)$$

EX: $4x^2 y'' + 8xy' + y = 0$

LET $y = x^m$: $4x^2\left(m(m-1)x^{m-2}\right) + 8x\left(mx^{m-1}\right) + x^m = 0$

$$\left(4m(m-1) + 8m + 1\right)x^m = 0$$

$$4m^2 + 4m + 1 = 0$$

$$4m^2 + 2m + 2m + 1 = 0$$

$$2m(2m+1) + (2m+1) = 0$$

$$(2m+1)^2 = 0$$

$$m = -\tfrac{1}{2}, -\tfrac{1}{2} \quad \text{CASE 2}$$

$$Y = c_1 x^{-\frac{1}{2}} + c_2 (\ln x) x^{-\frac{1}{2}}$$

EX: $x^2 y'' + 3xy' + 5y = 0$

LET $y = x^m$: $m(m-1) + 3m + 5 = 0$

$$m^2 + 2m + 5 = 0$$

$$m = \frac{1}{2}\left(-2 \pm \sqrt{4 - 20}\right)$$

$$m = \frac{1}{2}\left(-2 \pm 4i\right)$$

$$m = -1 \pm 2i \quad \text{CASE 3}$$

HERE $\alpha = -1$, $\beta = 2$ SO OUR GENERAL SOLUTION IS:

$$Y = C_1 X^{-1} \cos(2\ln x) + C_2 X^{-1} \sin(2\ln x)$$