LAST TIME: METHOD OF UNDETERMINED COEFF. - WAY TO FIND Y_P IN AN EQ. OF THE FORM:

$$ay''+by'+cy=g$$

FOR $a,b,c \in \mathbb{R}$ AND SOME FUNCTION $g(x)$

NAMELY, GUESS $Y_P = ?$ BASED ON (1) WHAT Y_H IS (WE DON'T GUESS ANYTHING IN Y_H)

(2) WHAT g, g', g'', g''' ETC. LOOK LIKE (GUESS ALL OF THOSE KINDS OF TERMS)

IF OUR GUESS REPEATS SOMETHING IN Y_H, WE MULTIPLY BY THE SMALLEST POWER OF X TO GET RID OF THE REPETITION.

ALSO, WE WENT OVER CAUCHY-EULER EQUATIONS (i.e. ONES WHERE $Y=X^m$ SUBSTITUTION WORKS)

FOR EXAMPLE, ANYTHING OF THE FORM:

$$ax^2y''+bxy'+cy=0$$

OR PERHAPS SOMETHING LIKE:

$$axy''+by'=0$$

SINCE WE CAN STILL FACTOR OUT ALL OF THE X'S ONCE WE PLUG-IN FOR Y.

DEALING W/ 2ND ORDER DIFFY Q'S, AGAIN IF WE ASSUME $Y=X^m$ WE WILL GET TWO ROOTS m_1 AND m_2 THAT WILL WORK, GIVING US 3 CASES:

CASE 1: $m_1 \neq m_2$ REAL ROOTS $\boxed{Y=c_1X^{m_1}+c_2X^{m_2}}$ GENERAL SOLUTION

CASE 2: $m_1=m_2$ REPEATED ROOTS $\boxed{Y=c_1X^{m_1}+c_2(\ln x)X^{m_1}}$

CASE 3: $m_1=\alpha+i\beta$, $m_2=\alpha-i\beta$ COMPLEX ROOTS $\boxed{Y=c_1X^{\alpha}\cos(\beta\ln x)+c_2X^{\alpha}\sin(\beta\ln x)}$

AS W/ THE SUBSTITUTION $Y=e^{mx}$ FOR 2ND ORDER DIFFY Q'S W/ CONSTANT COEFFICIENTS, THIS METHOD CAN BE APPLIED TO HIGHER ORDER EQ'S AS LONG AS IF WE GET ROOTS OF MULTIPLICITY GREATER THAN 2 WE INCLUDE MORE "MULTIPLY BY $(\ln x)$" TERMS. FOR EXAMPLE IF WE HAVE SOMETHING LIKE: $ax^3y'''+bx^2y''+cxy'+dy=0$ AND PLUG IN $Y=X^m$ AND GET $(m-2)^3=0$ OUR GEN. SOL. WOULD BE:

$$\boxed{Y=c_1X^2+c_2(\ln x)X^2+c_3(\ln x)^2X^2}$$

EX: $x^3 y''' - 6y = 0$ LET $y = x^m$ $y'' = m(m-1)x^{m-2}$

$y' = mx^m$ $y''' = m(m-1)(m-2)x^{m-3}$

PLUG IN: $x^3\, m(m-1)(m-2)\, x^{m-3} - 6x^m = 0$

$$\underbrace{\left[m(m-1)(m-2) - 6\right]}\, x^m = 0$$

$$m^3 - 3m^2 + 2m - 6 = 0$$

$$(m^2+2)(m-3) = 0$$

(USE RATIONAL ROOT TEST TO FIND $m=3$ IS A ROOT)

$m = 3$ OR $m^2 = -2$

$m = \pm\sqrt{2}\, i$ $\begin{pmatrix} \alpha = 0 \\ \beta = \sqrt{2} \end{pmatrix}$

SO THEN OUR GENERAL SOLUTION IS:

$$y = c_1 x^3 + c_2 \cos(\sqrt{2} \ln x) + c_3 \sin(\sqrt{2} \ln x)$$

WITH THESE TECHNIQUES WE WILL LOOK AT HARMONIC MOTION OF A SPRING AND SHOW ONE APPLICATION OF THESE METHODS.

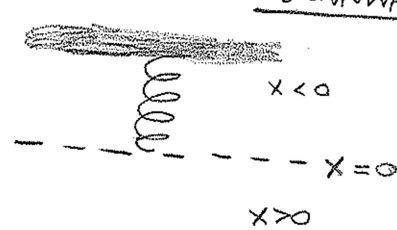
FIRST:

HOOKE'S LAW ANY SPRING HAS A "SPRING CONSTANT" DENOTED k DESCRIBING HOW HARD IT IS TO STRETCH OR COMPRESS. THE FORCE OF REPULSION IS GIVEN BY:

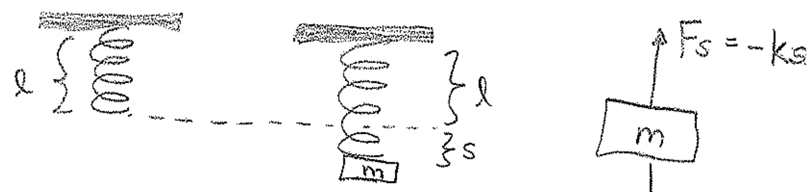
$$F_s = -kx \quad (<0 \text{ IMPLIES IT IS DIRECTED TOWARDS } x=0)$$

WHERE x = DISPLACEMENT OF THE SPRING END. k IS ALWAYS >0.

NOTE: WE WILL ALWAYS ASSUME THAT POSITIVE x MEANS STRETCHING DOWNWARD SINCE THE BOOK USES THIS ORIENTATION.



SUPPOSE WE HAVE A SPRING OF LENGTH ℓ AND WE ATTACH A MASS OF MASS m TO IT AND ALLOW THE SYSTEM TO COME TO REST:



AT THIS POINT THERE ARE TWO FORCES ACTING ON THE MASS, F_s (SPRING FORCE) & F_g (GRAVITY) AND BY NEWTON'S 2ND LAW:

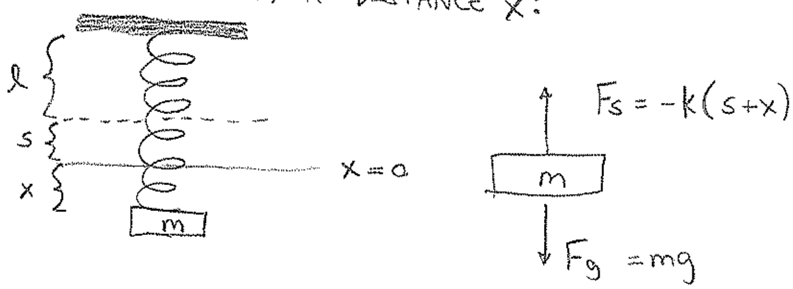
$$\underbrace{\sum F}_{\text{NET FORCE}} = \underbrace{m a}_{\text{MASS}\times\text{ACCELERATION}} \longrightarrow F_g + F_s = m\frac{dx^2}{dt^2}$$

$$mg - ks = 0 \qquad \left(\frac{d^2x}{dt^2} = 0 \text{ SINCE THE MASS IS AT REST}\right)$$

$$mg = ks$$

WE CALL THIS REST POINT THE <u>EQUILIBRIUM POSITION</u>, AND CALL THIS $x = 0$.

NOW WE WISH TO DESCRIBE THE MOTION OF THE MASS IF WE MOVE IT FROM THE EQUILIBRIUM POSITION BY A DISTANCE x:



BY NEWTON'S 2ND LAW: $$mg - k(s+x) = m\frac{d^2x}{dt^2}$$

$$\underbrace{(mg - ks)}_{\substack{=0 \\ \text{BY ABOVE}}} - kx = m\frac{d^2x}{dt^2}$$

$$0 = \frac{d^2x}{dt^2} + \frac{k}{m}x$$

WE TYPICALLY WRITE $\omega^2 = \frac{k}{m}$

$$0 = \frac{d^2x}{dt^2} + \omega^2 x$$

THE GENERAL SOLUTION CAN BE FOUND VIA $x = e^{mt}$ (DIFFERENT m NOT A MASS!)

$$0 = (m^2 + \omega^2)e^{mt}$$

$$m^2 = -\omega^2$$

$$m = \pm i\omega$$

SO $$x = c_1 \cos\omega t + c_2 \sin \omega t$$

BEFORE OUR 1ST EXAMPLE A REMINDER ON UNITS:

WEIGHT: mg UNITS: N (NEWTONS) OR lbs
(FORCE)

MASS: m UNITS: kg OR slugs

USE THIS

$g = 9.8\ m/s^2 \sim 10\ m/s^2$

$32\ ft/s^2$

EX: A FORCE OF 128 lbs STRETCHES A SPRING 4 ft. SUPPOSE WE ATTACH A MASS OF 2 slugs TO THIS SPRING AND RELEASE IT FROM REST 2ft BELOW THE EQUILIBRIUM POSITION. THEN FIND THE EQ. OF MOTION.

THE FIRST SENTENCE LET'S US FIND k BY HOOKE'S LAW:

$$F_s = kx$$

$$128 = 4k$$

$$32\ \text{lbs/ft} = k$$

NOTE: WE DON'T CARE ABOUT THE SIGN SINCE $k>0$ ALWAYS

NOW OUR EQ. OF MOTION IS: $x'' + \frac{k}{m}x = 0 \longrightarrow \underline{m = 2\text{ slugs}}$

$$x'' + \frac{32}{2}x = 0$$

$$x'' + 16x = 0$$

$\omega^2 = 16$ SO OUR GEN. SOL. IS:

$$x = c_1 \cos 4t + c_2 \sin 4t, \quad x' = -4c_1 \sin 4t + 4c_2 \cos 4t$$

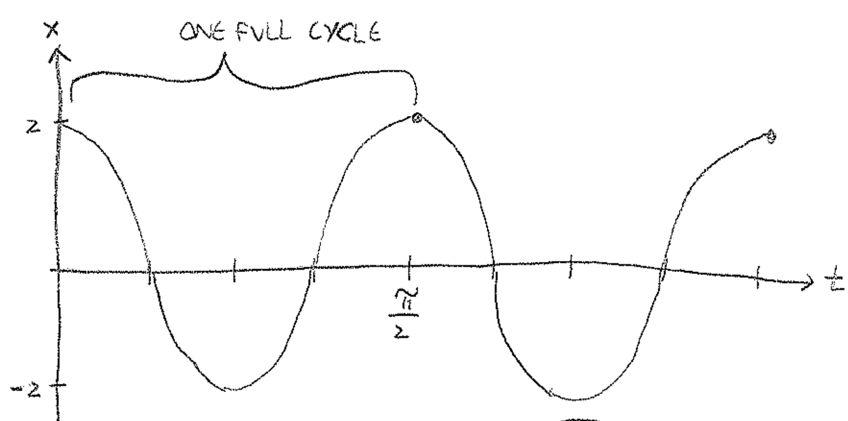
NOW WE FIND c_1 & c_2. "STARTING AT REST" IMPLIES THAT $x'(0) = 0$.

"STARTING 2ft BELOW EQUIL." IMPLIES THAT $x(0) = 2$ (RECALL $\downarrow$ IS POSITIVE x)

SO: $x(0) = c_1 = 2$

$x'(0) = 4c_2 = 0$, $c_2 = 0$

AND THUS: $\boxed{x = 2\cos 4t}$



IF WE LOOK AT THE GRAPH, THE TIME IT TAKES FOR x TO GO THROUGH A FULL CYCLE IS CALLED THE PERIOD, USUALLY DENOTED T.

IN OUR CASE, $T = \frac{\pi}{2}$. IN GENERAL, THE PERIOD OF $\cos \omega t$ IS $T = \frac{2\pi}{\omega}$. THIS IS BECAUSE

THE FREQUENCY IS DEFINED AS $f = \frac{1}{T}$ (i.e. CYCLES PER SECOND)

AT TIME $t_0 = \frac{2\pi}{\omega}$, $\cos \omega t_0 = \cos \omega\left(\frac{2\pi}{\omega}\right) = \cos 2\pi$. (i.e. ONE FULL CYCLE)

WE CAN COMBINE SOL'S OF THE FORM:

$$X = 3\cos 2t - 4\sin 2t$$

INTO ONE SINE CURVE: $X = A\sin(2t+\phi)$ FOR SOME A & ϕ.

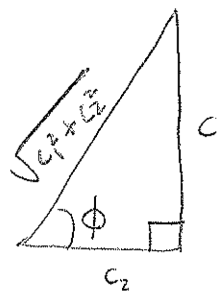
"AMPLITUDE" (A) "PHASE ANGLE" (ϕ)

THIS IS DONE USING THE FORMULA:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A \quad \star$$

SUPPOSE WE START W/ $X = c_1\cos\omega t + c_2\sin\omega t$

CONSIDER THE Δ:



SO THEN $\sin\phi = \dfrac{c_1}{\sqrt{c_1^2+c_2^2}} \longrightarrow c_1 = \sqrt{c_1^2+c_2^2}\,\sin\phi$

$\cos\phi = \dfrac{c_2}{\sqrt{c_1^2+c_2^2}} \longrightarrow c_2 = \sqrt{c_1^2+c_2^2}\,\cos\phi$

SUBBING THESE IN:

$$X = \sqrt{c_1^2+c_2^2}\left[\sin\phi\cos\omega t + \cos\phi\sin\omega t\right]$$

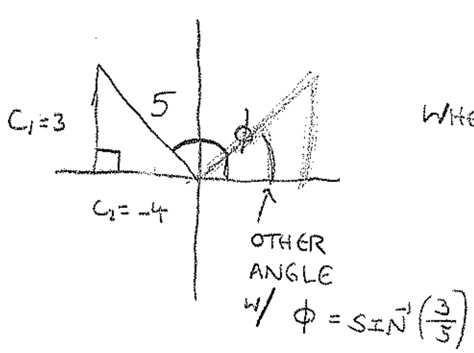
$$X = \sqrt{c_1^2+c_2^2}\,\sin(\omega t+\phi) \quad \text{BY } \star$$

BE CAREFUL WHEN CHOOSING ϕ. IF YOU ONLY LOOK AT $\phi = \sin^{-1}\left(\dfrac{c_1}{\sqrt{c_1^2+c_2^2}}\right)$ THERE ARE TWO ANGLES THAT WILL WORK. YOU NEED TO LOOK AT THIS AND $\phi = \cos^{-1}\left(\dfrac{c_2}{\sqrt{c_1^2+c_2^2}}\right)$.

EX: CONVERT $X = 3\cos 2t - 4\sin 2t$ TO A SINGLE TERM USING A, ϕ.

$c_1 = 3$, $c_2 = -4$ SO: $X = \sqrt{3^2+(-4)^2}\,\sin(2t+\phi)$

$= 5\sin(2t+\phi)$



WHERE $\phi = \sin^{-1}\left(\frac{3}{5}\right)$ ANGLE IN THE 2ND QUADRANT

$\approx \pi - .643 = 2.5$ RAD

$$X = 5\sin(2t + 2.5)$$

THIS CAN BE USEFUL WHEN FINDING SAY "FOR WHAT TIMES DOES $X=3$?" SINCE WE COMBINED TO ONE TRIG TERM.

DAMPING FORCES

SUPPOSE THERE IS THE SAME SPRING MOTION BUT IT IS RESISTED BY SOME SORT OF FORCE (SAY FRICTION) IN AN AMOUNT PROPORTIONAL TO THE VELOCITY x'. THEN WE CALL THIS A <u>DAMPING FORCE</u> $F_D = -\beta x'$ $(\beta > 0)$

HERE THE SIGN INDICATES THAT THE FORCE <u>OPPOSES THE DIRECTION OF MOTION</u>.

WE CALL β THE <u>DAMPING CONSTANT</u>. NOW WE APPLY NEWTON'S 2ND LAW:

$$\sum F = m\frac{d^2x}{dt^2}$$

$$\underbrace{mg}_{F_g} - \underbrace{k(s+x)}_{F_s} - \underbrace{\beta\frac{dx}{dt}}_{F_D} = m\frac{d^2x}{dt^2} \quad \text{(AGAIN } mg - ks = 0 \text{ CANCELS)}$$

$$-kx - \beta\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

SO $$0 = \frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x$$

LET $\omega^2 = \frac{k}{m}$ AND $\frac{\beta}{m} = 2\lambda$: (FOR CONVENIENCE LATER)

$$0 = \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x$$

AGAIN WE KNOW HOW TO FIND THE GENERAL SOLUTION TO THIS ($x = e^{mt}$ m IS <u>NOT</u> A MASS)

$$0 = \underbrace{(m^2 + 2\lambda m + \omega^2)}_{=0} e^{mt}$$

$$m = \frac{1}{2}\left(-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}\right)$$

$$m = \frac{1}{2}\left(-2\lambda \pm 2\sqrt{\lambda^2 - \omega^2}\right)$$

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \quad \text{CALL THEM } m_1 \text{ \& } m_2$$

AGAIN WE HAVE <u>3 CASES</u>

CASE 1 $\lambda^2 - \omega^2 > 0$, DISTINCT REAL $m_1 \neq m_2$

$$x = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

BOTH m_1 AND $m_2 < 0$ IN THIS CASE:

SINCE $\lambda = \frac{\beta}{2m}$, $\lambda > 0$.

SO OBVIOUSLY $m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2} < 0$. (BOTH <0 TERMS)

$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2} < 0$?

$\sqrt{\lambda^2 - \omega^2} < \lambda$ SQUARE BOTH SIDES

$\cancel{\lambda^2} - \omega^2 < \cancel{\lambda^2}$

$-\omega^2 < 0$ YES.

SO BOTH TERMS ARE EXPONENTIAL DECAY. THIS IS BECAUSE $\lambda^2 > \omega^2$ i.e. λ IS LARGE (WHICH MEANS β IS LARGE) AND THUS THE DAMPING FORCE WAS GREAT ENOUGH TO CANCEL OUT ANY POSSIBLE OSCILLATIONS. THIS IS CALLED OVERDAMPED.

CASE 2 $\lambda^2 - \omega^2 = 0$, REPEATED REAL $m_1 = m_2 = -\lambda < 0$

THEN $\boxed{y = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}}$

THIS SITUATION IS CALLED CRITICALLY DAMPED

CASE 3 $\lambda^2 - \omega^2 < 0$, COMPLEX ROOTS

$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

$m = -\lambda \pm \sqrt{(-1)(\omega^2 - \lambda^2)}$

$m = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$ SO $\alpha = -\lambda$, $\beta = \sqrt{\omega^2 - \lambda^2}$

SO $\boxed{y = c_1 e^{-\lambda t} \cos\left(\sqrt{\omega^2 - \lambda^2}\, t\right) + c_2 e^{-\lambda t} \sin\left(\sqrt{\omega^2 - \lambda^2}\, t\right)}$

THIS SITUATION IS CALLED UNDERDAMPED SINCE THE SPRING WILL STILL OSCILLATE BUT WITH DECAYING AMPLITUDE.

WE WILL USE THESE CALCULATIONS TO QUICKLY ARRIVE AT GENERAL SOLUTIONS TO SPRING PROBLEMS. THERE IS NO NEED TO MEMORIZE THESE CASES SINCE WE REALLY ALREADY KNOW HOW TO DO THESE PROBLEMS — OUR SUBSTITUTIONS FOR λ & ω^2 JUST MAKE IT SIMPLER TO STATE.

EX: A MASS WEIGHING 4 lbs IS ATTACH TO A SPRING W/ $k = 2$ lbs/ft. THE MEDIUM OFFERS A DAMPING FORCE NUMERICALLY EQUAL TO THE INSTANTANEOUS VELOCITY. THE MASS IS INITIALLY RELEASED FROM A POINT 1 FT ABOVE EQUILIBRIUM W/ A DOWNWARD VELOCITY OF 8 ft/s. FIND THE EQ. OF MOTION.

WE HAVE: $m x'' + \beta x + k x = 0$

THE MASS <u>WEIGHS</u> 4 lbs SO $mg = 4$

$$m = \frac{4}{g} = \frac{4 \text{ lbs}}{32 \text{ ft/s}^2} = \frac{1}{8} \text{ slugs}$$

$\beta = 1$ BY SENTENCE #2.

$k = 2$ lbs/ft IS GIVEN.

$$\frac{1}{8} x'' + x' + 2x = 0$$

DIVIDING BY $\frac{1}{8}$ WE GET: $x'' + \underset{2\lambda}{\underset{\uparrow}{8}}x' + \underset{\omega^2}{\underset{\uparrow}{16}}\, x = 0$

SO $\lambda = 4, \quad \omega = 4$

AND $\underline{\lambda^2 - \omega^2 = 0}$ CASE 2 "CRITICALLY DAMPED" SO WE HAVE OUR SOLUTION IS:

$$y = c_1 e^{-4t} + c_2 t e^{-4t} \qquad y' = -4c_1 e^{-4t} + c_2\left(e^{-4t} - 4te^{-4t}\right)$$

$y(0) = -1 = c_1$ (1 FT ABOVE EQUILIBRIUM)

$y'(0) = 8 = -4c_1 + c_2$

$4 = c_2$

SO $y = -e^{-4t} + 4te^{-t}$

EX: A 16 lb WEIGHT IS ATTACHED TO A 5 ft SPRING. AT EQUILIBRIUM THE SPRING MEASURES 8.2 ft. SUPPOSE THE WEIGHT IS RELEASED FROM REST AT A POINT 2 ft ABOVE EQUIL. AND SUPPOSE THERE IS A DAMPING NUMERICALLY EQUIVALENT TO THE INSTANTANEOUS VELOCITY. FIND THE EQUATION OF MOTION.

WE FIND THE MASS OF THE WEIGHT:

$$mg = 16 \text{ lb}$$

$$m = \frac{16 \text{ lb}}{32 \text{ ft/s}^2} = \frac{1}{2} \text{ slug}$$

NOW FIND k (FORCE OF 16 lbs STRETCHED IT $8.2 - 5 = 3.2$ ft)

$$F = kx$$

$$16 = k(3.2)$$

$$5 \text{ lb/ft} = k$$

AND $\beta = 1$

SO OUR EQ IS: $\frac{1}{2}x'' + x' + 5x = 0$

$$\underbrace{x'' + 2x'}_{2\lambda} + \underbrace{10x}_{\omega^2} = 0$$

SO $\lambda = 1$, $\omega = \sqrt{10}$

$\lambda^2 - \omega^2 = 1 - 10 = -9 < 0$ CASE 3 OVERDAMPED

$$x = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$$

OUR INITAL CONDITIONS ARE $x(0) = 2$, $x'(0) = 0$ ("FROM REST")

THIS SHOULD GET US $c_1 = -2$, $c_2 = -\frac{2}{3}$ SO

$$x = -2e^{-t}\cos 3t - \frac{2}{3}e^{-t}\sin 3t$$

NOW, SUPPOSE WE ADD ANOTHER FORCE INTO THE SYSTEM, (FOR EXAMPLE THE THING THE SPRING IS ATTACHED TO STARTS MOVING). THEN IF WE MODEL THIS OTHER FORCE BY A FUNCTION $f(t)$, NEWTON'S LAW READS:

$$\sum F = mx''$$

$$mg - k(s+x) - \beta x' + f(t) = mx'' \qquad (mg = ks)$$

$$f(t) = mx'' + \beta x' + kx$$

IN OTHER WORDS, SOLVING FOR THE EQUATION OF MOTION REDUCES TO SOLVING THE NONHOMOG. DIFFY Q ABOVE (WHICH WE CAN DO W/ UNDETERMINED COEFFS USUALLY).

EX: <u>RESONANCE FREQUENCIES.</u>

SUPPOSE WE HAVE A SPRING IN MOTION W/ NO DAMPING SO THAT OUR EQ. OF MOTION IS SINUSOIDAL, BUT WE THEN APPLY A FORCE WITH A SINUSOIDAL PATTERN AS WELL. IF THE FREQUENCIES OF MOTION AND THE APPLIED FORCE ARE CLOSE, WE SHOULD EXPECT OUR SPRING/MASS SYSTEM TO HAVE VERY LARGE AMPLITUDES. SO LET'S SUPPOSE WE HAVE:

$$x'' + \omega^2 x = F_0 \sin(\gamma t) \qquad \underbrace{x(0)=0, \quad x'(0)=0}_{\text{MASS STARTS AT REST FROM EQUILIBRIUM}}$$

↑ SINUSOIDAL FORCE APPLIED

$\gamma, F_0 =$ CONSTANTS

SUPPOSE $\omega \neq \gamma$

LET'S FIND THE EQ. OF MOTION:

FIRST WE FIND THE GENERAL SOLUTION $x = x_p + x_H$

x_H: $m^2 + \omega^2 = 0$ IF $x = e^{mt}$ (m IS NOT A MASS!)

$m^2 = -\omega^2$

$m = \pm i\omega$

$$x_H = c_1 \cos \omega t + c_2 \sin \omega t$$

FOR x_p, WE GUESS $x_p = A\cos\gamma t + B \sin \gamma t$ (SINCE $\gamma \neq \omega$, THESE ARE NOT IN x_H!)

$x_p' = -A\gamma \sin\gamma t + B\gamma \cos\gamma t$

$x_p'' = -A\gamma^2 \cos\gamma t - B\gamma^2 \sin\gamma t$

PLUG IN: $x_p'' + \omega^2 x_p = (-\gamma^2+\omega^2)A\cos\gamma t + (-\gamma^2+\omega^2)B\sin\gamma t = F_0 \sin\gamma t$

SO $A = 0$ AND $B = \dfrac{F_0}{\omega^2-\gamma^2}$

THUS THE GENERAL SOLUTION IS:

$$X = X_P + X_H$$

$$X = \frac{F_0}{\omega^2-\gamma^2}\sin\gamma t + c_1\cos\omega t + c_2\sin\omega t$$

$$X(0) = c_1 = 0 \qquad \text{so} \quad X = \frac{F_0}{\omega^2-\gamma^2}\sin\gamma t + c_2\sin\omega t$$

$$X' = \frac{\gamma F_0}{\omega^2-\gamma^2}\cos\gamma t + c_2\omega\cos\omega t$$

$$X'(0) = 0 = \frac{\gamma F_0}{\omega^2-\gamma^2} + c_2\omega$$

$$\frac{-\gamma F_0}{\omega(\omega^2-\gamma^2)} = c_2 \quad , \quad c_2 = \frac{\gamma F_0}{\omega(\gamma^2-\omega^2)}$$

SO OUR EQ. OF MOTION IS:

$$X = \frac{F_0}{\omega^2-\gamma^2}\sin\gamma t + \frac{\gamma F_0}{\omega(\gamma^2-\omega^2)}\sin\omega t$$

$$X = \frac{F_0}{\omega(\omega^2-\gamma^2)}\left(\omega\sin\gamma t - \gamma\sin\omega t\right)$$

NOW NOTICE $\frac{F_0}{\omega(\omega^2-\gamma^2)} \longrightarrow \infty$ AS $\gamma \to \omega$ BUT THE SECOND TERM GOES TO ZERO.

OUR INTUITION TELLS US THAT THIS PRODUCT SHOULD BECOME AN UNBOUNDED FUNCTION AS $\gamma \to \omega$, SO LET'S USE L'HÔPITAL'S RULE:

$$\lim_{\gamma\to\omega} X = \lim_{\gamma\to\omega} \frac{F_0(\omega\sin\gamma t - \gamma\sin\omega t)}{\omega(\omega^2-\gamma^2)}$$

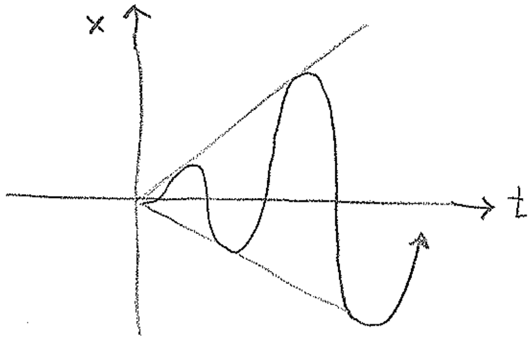
TOP & BOTTOM $\longrightarrow 0$
SO TAKE DERIVATIVES:
(WITH RESPECT TO γ)

$$= \frac{\lim_{\gamma\to\omega} \frac{d}{d\gamma}\left(F_0(\omega\sin\gamma t - \gamma\sin\omega t)\right)}{\lim_{\gamma\to\omega} \frac{d}{d\gamma}\left(\omega(\omega^2-\gamma^2)\right)}$$

UNBOUNDED

$$= \frac{\lim_{\gamma\to\omega} F_0(\omega t\cos\gamma t - \sin\omega t)}{\lim_{\gamma\to\omega} -2\omega\gamma} = \frac{F_0(\omega t\cos\omega t - \sin\omega t)}{-2\omega^2} = \frac{F_0\sin\omega t}{2\omega^2} - \frac{F_0\omega t\cos\omega t}{2\omega^2}$$

THE GRAPH OF THIS X AS A FUNCTION OF t IS SOMETHING LIKE:



THIS PHENOMENON IS KNOWN AS PURE RESONANCE. IT IS UNREALISTIC THAT WE ASSUME THAT THERE ARE NO DAMPING FORCES, HOWEVER.

ONE MORE EXAMPLE:

EX: A MASS OF 1 slug STRETCHES A SPRING 2ft AND THEN COMES TO REST AT EQUILIBRIUM. AN EXTERNAL FORCE $f(t) = 8\sin 4t$ IS THEN APPLIED. FIND THE EQ. OF MOTION IF THE DAMPING FORCE IS NUMERICALLY EQUAL TO 8 TIMES THE INSTANTANEOUS VELOCITY.

THEY TELL US $m=1$, $\beta=8$. WE USE SENTENCE #1 TO FIND k:

$$F = kx$$

$$mg = 2k$$

$$\frac{(1)(32)}{2} = k = 16 \text{ lbs/ft}$$

SO THEN: $mx'' + \beta x' + kx = 8\sin 4t = f(t)$

$$x'' + 8x' + 16x = 8\sin 4t$$

FIRST WE FIND X_H: $m^2 + 8m + 16 = 0$ IF $x = e^{mt}$

$$(m+4)^2 = 0$$

$$m = -4, -4 \qquad \boxed{X_H = c_1 e^{-4t} + c_2 t e^{-4t}}$$

NOW WE GUESS $X_P = A\cos 4t + B\sin 4t$, FIND $A = -\frac{1}{4}$, $B = 0$

SO $X = -\frac{1}{4}\cos 4t + c_1 e^{-4t} + c_2 t e^{-4t}$ IS THE GENERAL SOLUTION.

SINCE OUR MASS STARTS AT REST $X'(0) = 0$ AND FROM THE EQUIL. POSITION $X(0) = 0$.

$X(0) = -\frac{1}{4} + c_1 = 0$ SO $c_1 = \frac{1}{4}$

$$X'(t) = \sin 4t - 4c_1 e^{-4t} + c_2(e^{-4t} - 4te^{-4t})$$

$$X'(0) = -4c_1 + c_2 = 0 \qquad \boxed{X = -\frac{1}{4}\cos 4t + \frac{1}{4}e^{-4t} + te^{-4t}}$$

$$c_2 = 4c_1 = 1$$