LAST TIME WE APPLIED OUR METHODS OF SOLVING DIFFY Q'S TO FIND EQUATIONS DESCRIBING THE MOTION OF A MASS ON A SPRING JUST BY USING NEWTON'S 2ND LAW, AND GETTING:

$$mx'' + \beta x' + kx = f(t)$$

WHERE $m$ = MASS OF OBJECT ON SPRING

$\beta$ = DAMPING CONSTANT

$k$ = SPRING CONSTANT

$f(t)$ = OTHER EXTERNAL FORCE OR "DRIVING FORCE"

NOW WE MOVE ON TO:

## SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

THESE ARE SYSTEMS OF DIFFY Q'S OF THE FORM: ($x$, $y$, AND $z$ ARE FUNCTIONS OF $t$)

$$\frac{dx}{dt} = 3x + 4y$$
$$\frac{dy}{dt} = x - y$$

(IN 2 VARIABLES)

OR

$$\frac{dx}{dt} = x - z$$
$$\frac{dy}{dt} = -y + 3z$$
$$\frac{dz}{dt} = x + y + z$$

(IN 3 VARS) ETC.

THESE SYSTEMS CAN ALSO BE WRITTEN:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

THE GENERAL SHORTHAND FOR THESE IS $X' = AX$ WHERE $X$ IS A VECTOR AND $A$ IS THE MATRIX OF COEFFICIENTS. IN OUR CASE, $A$ WILL ALWAYS BE A MATRIX OF CONSTANTS BUT A GENERAL SYSTEM OF LINEAR DIFFY Q'S COULD HAVE FUNCTIONS INSIDE $A$. ANALOGOUSLY, A SYSTEM OF THE FORM

$X' = AX$ (OR $X' - AX = 0$) IS CALLED HOMOGENEOUS.

AND ONE OF THE FORM

$X' = AX + V$ (OR $X' - AX = V$) WHERE $V$ IS A VECTOR OF FUNCTIONS IS CALLED NONHOMOGENEOUS.

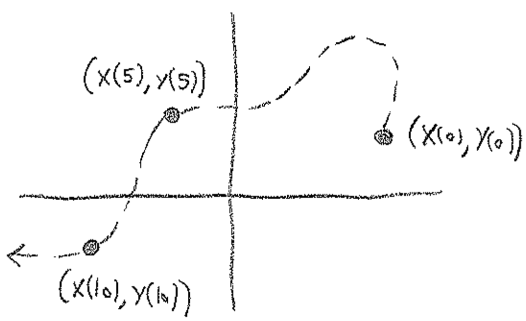
WE WILL ONLY DEAL WITH HOMOGENEOUS SYSTEMS. SOLVING NONHOMOGENEOUS SYSTEMS CAN BE DONE USING METHODS SIMILAR TO UNDETERMINED COEFFICIENTS AND VARIATION OF PARAMETERS (SECT. 3.5 NOT ON OUR SYLLABUS).

EX: $$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3t - \cos t \\ \sin t \end{pmatrix}$$

IS NONHOMOGENEOUS

OUR "V" IN $X' = AX + V$

RMK: SO WHEN GIVEN A SYSTEM AS ABOVE, WE ARE LOOKING FOR FUNCTIONS $X(t)$ AND $Y(t)$ SATISFYING SOME CONDITIONS ON THEIR DERIVATIVES. ONCE WE PICK OUT INITIAL VALUES $X(0)$ AND $Y(0)$ OUR SOLUTION CAN BE THOUGHT OF AS A PATH IN $\mathbb{R}^2$ STARTING AT THE POINT $(X(0), Y(0))$:




GRAPH OF $(X(t), Y(t))$

THIS WAY OF THINKING OF SOLUTIONS AS PATHS ONCE GIVEN AN INITIAL POINT IS IMPORTANT.

THM: WHEN SOLVING A NONHOMOGENEOUS SYSTEM

$$\underbrace{A}_{n \times n}\quad X' = AX + V \text{ WITH INITIAL CONDITIONS } X(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

OUR "INITIAL POINT" IN $\mathbb{R}^n$

WITH $A$ AND $V$ MATRICES OF CONTINUOUS FUNCTIONS OF $t$,

(IN OUR CASE $A$ IS A MATRIX OF CONSTANT FUNCTIONS AND $V = 0$)

THEN: 1. A SOLUTION EXISTS FOR $X$

2. THE SOLUTION IS <u>UNIQUE</u>

AGAIN THE UNIQUENESS WILL BE KEY IN SHOWING THAT ANY SOLUTION LOOKS LIKE A LINEAR COMBINATION OF A FEW SOLUTIONS.

LET'S JUST DO A QUICK EXAMPLE TO FAMILIARIZE OURSELVES WITH THESE IDEAS

EX: $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ AND $X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ ARE BOTH SOLUTIONS TO THE SYSTEM:

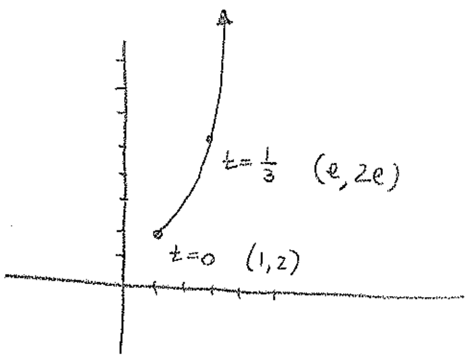
$$X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$$

CHECK $X_1$: $X_1' = \begin{pmatrix} 1 \\ 2 \end{pmatrix} 3e^{3t} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} e^{3t}$

EQUAL ✓

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X_1 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} e^{3t}$$

NOW LET'S PLOT $X_1$ AND VIEW IT AS THE MOTION OF A PARTICLE AS TIME $t$ VARIES:




ALSO NOTE THAT BOTH $X_1 + X_2$ AND $5X_1$ WILL BE SOLUTIONS TO THIS SYSTEM. THIS BRINGS US TO:

THM: THE SOLUTIONS TO A HOMOGENEOUS SYSTEM $X' = AX$ FORM A VECTOR SPACE.

PF: WE JUST NEED TO SHOW THAT FOR TWO SOLUTIONS $X_1$ & $X_2$,

1. $X_1 + X_2$ IS A SOLUTION
2. $cX_1$ IS A SOLUTION FOR ANY $c \in \mathbb{R}$
3. $0$ IS A SOLUTION

① $(X_1 + X_2)' = X_1' + X_2' = AX_1 + AX_2 = A(X_1 + X_2)$ ✓

② $(cX_1)' = cX_1' = cAX_1 = A(cX_1)$ ✓

③ LET $c = 0$ AND USE ②

THIS IS JUST LIKE OUR HOMOGENEOUS SOLUTIONS FOR THE SINGLE DIFFY Q'S WE DID EARLIER - WE CAN ADD AND SCALE SOLUTIONS TOGETHER TO GET ANOTHER ONE. NOW THE QUESTION IS THIS: HOW MANY SOLUTIONS DO WE NEED TO SPAN ALL OF THEM?

**DEF:** LET $X_1 \ldots X_k$ BE SOLUTIONS (VECTORS) OF THE SYSTEM $X' = AX$ ($A_{n \times n}$)

WE SAY THEY ARE L.I. IF $c_1 X_1 + c_2 X_2 + \ldots + c_k X_k = 0$ IMPLIES THAT

$$c_1 = c_2 = \ldots = c_k = 0$$

(EQUIVALENTLY, NO $X_i$ IS A LINEAR COMBINATION OF THE OTHER $X_i'$s)

AGAIN WE CAN USE WHAT WE WILL ALSO CALL THE WRONSKIAN TO TEST FOR L.I.:

DEF: IF WE HAVE $X_1 \ldots X_n$ SOLUTIONS TO THE SYSTEM $X' = AX$ ($A_{n \times n}$), THEN THEY ARE L.I. IFF THEIR WRONSKIAN:

$$W(X_1, \ldots, X_n) = \underbrace{\begin{vmatrix} X_1 & X_2 & \cdots & X_n \end{vmatrix}}_{\text{COLUMNS ARE THE VECTORS } X_i} \neq 0$$

FOR ANY $t$ ON THE INTERVAL ON WHICH WE'RE TESTING FOR L.I.

EX: OUR SOLUTIONS $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$, $X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ OF $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$ WERE L.I. ON ALL OF $\mathbb{R}$:

$$W(X_1, X_2) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0 \quad \underline{\text{FOR ANY } t}$$

AGAIN, THIS WILL IMPLY THAT <u>ANY</u> SOLUTION IS OF THE FORM $c_1 X_1 + c_2 X_2$!!

THM: IF $X_1, \ldots, X_n$ ARE L.I. SOLUTIONS TO $X' = AX$ THEN ANY OTHER SOLUTION IS OF THE FORM:

NEED $n$ OF THEM

$$X = c_1 X_1 + \ldots + c_n X_n$$

IN THIS CASE $X_1, \ldots, X_n$ ARE OFTEN CALLED A <u>FUNDAMENTAL SET</u> OF SOLUTIONS.

AGAIN THIS ALLOWS US TO FIND A GENERAL SOLUTION BY ONLY FINDING A FEW HOMOGENEOUS SOLUTIONS WHICH ARE L.I., SO LET'S MOVE ON TO TECHNIQUES TO FIND THESE $X_i$'s.

# HOMOGENEOUS LINEAR SYSTEMS

SUPPOSE WE WANT TO FIND A SOLUTION TO

$$X' = AX \quad (A \text{ IS A MATRIX OF CONSTANTS})$$

SOME SMART PERSON ONE DAY THOUGHT THAT A SOLUTION $X$ COULD BE SOMETHING OF THE FORM: $X = e^{\lambda t} V$ WHERE $V$ IS AN <u>EIGENVECTOR</u> OF $A$ WITH <u>EIGENVALUE</u> $\lambda$.

WHY WILL THIS WORK?

RECALL THE DEFINITION OF AN EIGENVECTOR:

① $V \neq 0$ ② $AV = \lambda V$ FOR SOME $\lambda \in \mathbb{R}$

① IMPLIES THAT OUR SOLUTION ISN'T TRIVIAL (MEANING $X =$ ZERO VECTOR)

$$X' = (e^{\lambda t} V)' \underset{\substack{\text{SINCE } V \text{ IS} \\ \text{A CONSTANT VECTOR}}}{=} (e^{\lambda t})' V = \lambda e^{\lambda t} V = e^{\lambda t}(\lambda V) \underset{\substack{\text{DEFINITION} \\ \text{PART ②}}}{=} e^{\lambda t} AV = A e^{\lambda t} V = AX$$

SO IT <u>IS A SOLUTION</u>. IT WORKS BECAUSE DIFFERENTIATING $e^{\lambda t} V$ AND MULTIPLICATION BY $A$ ARE BOTH REALLY JUST MULTIPLICATION BY $\lambda$.

<u>SO</u> IN GENERAL SUPPOSE WE ARE SOLVING A SYSTEM $X' = AX$, AND $A$ HAS $n$ L.I. EIGENVECTORS $V_1, \dots, V_n$ W/ EIGENVALUES $\lambda_1, \dots, \lambda_n$ CORRESPONDING TO THEM (THE $\lambda$'S NEED NOT BE DISTINCT). THEN BY ABOVE,

$$X_1 = e^{\lambda_1 t} V_1, \quad X_2 = e^{\lambda_2 t} V_2, \quad \dots, \quad X_n = e^{\lambda_n t} V_n$$

ARE ALL SOLUTIONS TO $X' = AX$. THEY ARE L.I. AS SOLUTIONS SINCE:

$$W(X_1, \dots, X_n) = \left| \begin{array}{c|c|c|c} e^{\lambda_1 t} V_1 & e^{\lambda_2 t} V_2 & \cdots & e^{\lambda_n t} V_n \end{array} \right| = \underbrace{e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t}}_{\neq 0} \underbrace{\left| \begin{array}{c|c|c|c} V_1 & V_2 & \cdots & V_n \end{array} \right|}_{\substack{\neq 0 \\ \text{SINCE } V\text{'S ARE} \\ \text{L.I.}}}$$

(FACTOR OUT $e^{\lambda_i t}$ FROM EACH COLUMN)

SO BY OUR THM, ANY SOLUTION OF $X' = AX$ IS OF THE FORM

$$X = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + \dots + c_n e^{\lambda_n t} V_n$$

THIS GIVES US THE GENERAL SOLUTION AS LONG AS WE HAVE $n$ L.I. EIGENVECTORS!

EX: FIND THE GENERAL SOLUTION OF:

$$X' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} X$$

WE FIND THE EIGENVALUES/VECTORS:

$$\begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) = 0$$

$$\lambda = -1, -4$$

$\lambda_1 = -1$, $\lambda_2 = -4$

DISTINCT $\lambda$'S $\Longrightarrow$ $n$ L.I. EIGENVECTORS SO WE KNOW THIS METHOD WILL WORK

EIGENVECTORS:

$V_1$: $(A + I) v_1 = 0$

$$\left(\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right) \quad R_1 \rightarrow R_1 + \sqrt{2} R_2 \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right) \quad \begin{array}{l} \sqrt{2}x - y = 0 \\ \sqrt{2}x = y \end{array} \quad \begin{pmatrix} x \\ \sqrt{2}x \end{pmatrix}$$

SO LET $V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

$V_2$: $(A + 4I) V_2 = 0$

$$\left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right) \quad R_2 \rightarrow R_2 - \sqrt{2} R_1 \quad \left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x + \sqrt{2}y = 0 \\ x = -\sqrt{2}y \end{array} \quad \begin{pmatrix} -\sqrt{2}y \\ y \end{pmatrix}$$

SO LET $V_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

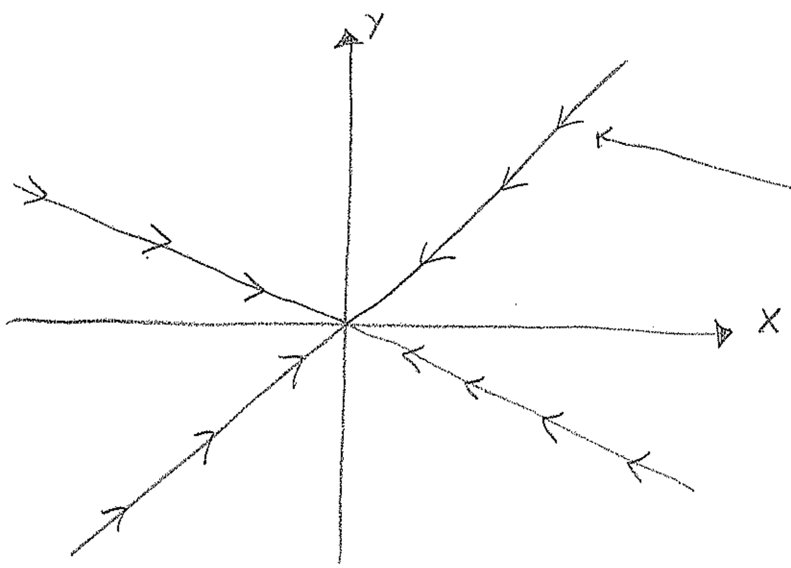
THE GENERAL SOLUTION IS THEN:

$$X = c_1 e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

LET'S LOOK AT OUR SOLUTION A BIT – WE CAN TELL A LOT ABOUT ANY SOLUTION JUST LOOKING AT OUR TWO WE FOUND.

$X_1 = e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ WAS ONE SOLUTION. NOTICE THAT AS $t$ VARIES OUR PATH IS CONFINED TO THE LINE CONTAINING TO POINT $(1, \sqrt{2})$. ALSO NOTE THAT THE $e^{-t}$ MAKES THE SOLUTIONS ALL CONVERGE TO $(0,0)$.

$X_2 = e^{-4t}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ HAS SIMILAR PROPERTIES.




HERE IS A GRAPH OF WHAT POSSIBLE PATHS LOOK LIKE. WE CAN CHANGE OUR STARTING POINT BY IMPOSING AN INITIAL CONDITION SAY $X(0) = \begin{pmatrix} 3 \\ 3\sqrt{2} \end{pmatrix}$

I CHOSE THIS POINT TO BE ON THE LINE

THEN WE FIND $c_1$ & $c_2$:

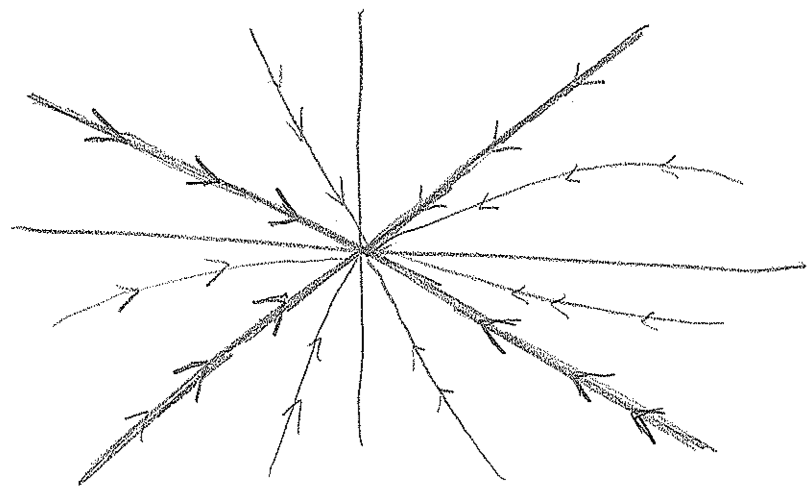
$$X(0) = c_1 \cancel{e^0}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 \cancel{e^0}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3\sqrt{2} \end{pmatrix}$$

$$c_1 - \sqrt{2}c_2 = 3$$
$$\sqrt{2}c_1 + c_2 = 3\sqrt{2} \qquad E_2 \to E_3 - \sqrt{2}E_1$$

$$c_1 - \sqrt{2}c_2 = 3$$
$$3c_2 = 0$$
$$c_2 = 0$$

AND THUS $c_1 = 3$

SO OUR SOLUTION IS $X(t) = 3e^{-t}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$. FOR THE SAME REASONS, IF WE START ANYWHERE ELSE ON EITHER OF THESE LINES CONTAINING THE EIGENVECTORS, OUR SOLUTION PATH WILL NOT LEAVE THEM! THE OTHER PATHS FIT IN NICELY:



THIS IS CALLED THE <u>PHASE PLANE</u>

ALL OF THE PATHS $\longrightarrow (0,0)$ SINCE BOTH TERMS IN THE GENERAL SOLUTION HAVE EXPONENTIAL DECAY.

EX: $X' = \begin{pmatrix}3 & 2\\ -5 & -4\end{pmatrix}X$ FIND GEN. SOL.

$\lambda$'s: $\begin{vmatrix}3-\lambda & 2\\ -5 & -4-\lambda\end{vmatrix} = (3-\lambda)(-4-\lambda)+10 = \lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) = 0$

$\lambda = 1, -2$

LET $\lambda_1 = 1$, $\lambda_2 = -2$

$(A-I)v_1 = 0$

$\left(\begin{array}{cc|c}2 & 2 & 0\\ -5 & -5 & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$ $\quad x+y=0$, $x=-y$ $\quad \begin{pmatrix}-y\\ y\end{pmatrix} \xrightarrow[y=1]{\text{CHOSE}} v_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$

$(A+2I)v_2 = 0$

$\left(\begin{array}{cc|c}5 & 2 & 0\\ -5 & -2 & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}5 & 2 & 0\\ 0 & 0 & 0\end{array}\right)$ $\quad 5x+2y=0$, $x = -\frac{2}{5}y$ $\quad \begin{pmatrix}-\frac{2}{5}y\\ y\end{pmatrix} \xrightarrow[y=5]{\text{CHOSE}} v_2 = \begin{pmatrix}-2\\ 5\end{pmatrix}$

$$X = c_1 e^{t}\begin{pmatrix}-1\\ 1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}-2\\ 5\end{pmatrix}$$

LET'S MAKE A SIMILAR PLOT:

SOLUTIONS OF FORM $c_1 e^{t}\begin{pmatrix}-1\\1\end{pmatrix}$ $(C_2=0)$

SOLUTIONS OF FORM $c_2 e^{-2t}\begin{pmatrix}-2\\5\end{pmatrix}$ $(C_1=0)$

NOTICE THAT OUR ONLY SOLUTIONS

$$c_1 e^{t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}-2\\5\end{pmatrix}$$

THAT ARE BOUNDED AS $t \to \infty$ ARE THE ONES WHERE $C_1 = 0$. THIS CAN BE SEEN IN OUR PLOT.

NOW WHAT HAPPENS IF WE HAVE COMPLEX $\lambda$'S AND $V$'S?

THE SAME GENERAL SOLUTION IS CORRECT (IF WE HAVE $n$ L.I. EIGENVECTORS) BUT AGAIN WE WANT REAL SOLUTIONS.

EX: FIND THE GEN. SOL. OF $X' = \begin{pmatrix}2&-1\\1&2\end{pmatrix}X$

$\lambda$'s: $\begin{vmatrix}2-\lambda & -1\\ 1 & 2-\lambda\end{vmatrix} = (2-\lambda)^2 + 1 = 0$

$$(2-\lambda)^2 = -1$$

$$2-\lambda = \pm i$$

$$\lambda = 2 \pm i$$

$\lambda_1 = 2+i, \quad \lambda_2 = \overline{\lambda_1}$

$V$'s: $(A-(2+i)I)V_1 = 0$

$$\left(\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right) \quad R_1 \to R_1 + iR_2 \quad \left(\begin{array}{cc|c}0 & 0 & 0\\ 1 & -i & 0\end{array}\right)$$

$x - iy = 0$

$x = iy$

LET $y = 1$

$$V_1 = \begin{pmatrix}i\\1\end{pmatrix}$$

$$V_2 = \overline{V_1} = \begin{pmatrix}-i\\1\end{pmatrix}$$

RECALL COMPLEX $\lambda$'S AND $V$'S COME IN CONJUGATE PAIRS

SO GEN. SOLUTION IS:

$$X = c_1 e^{(2+i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{(2-i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

BUT HOW DO WE PICTURE SOLUTIONS IN THE PHASE PLANE? WE NEED TO GET RID OF THE $i$'S. WE DO THIS IN GENERAL NOW:

SUPPOSE $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ ARE EIGENVALUES W/ EIGENVECTORS

$V_1 = B_1 + iB_2$, $V = B_1 - iB_2$ WHERE $B_1$ & $B_2$ ARE REAL VECTORS

EX: $\begin{pmatrix} 1-i \\ 3+2i \end{pmatrix} = \underset{\uparrow \atop B_1}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} + i\underset{\uparrow \atop B_2}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

AGAIN WE APPLY EULER'S FORMULA:

OUR SOL'S ARE $X_1 = e^{\lambda_1 t}V_1$, $X_2 = e^{\lambda_2 t}V_2 = \overline{X_1}$

$$X_1 = e^{(\alpha+i\beta)t}(B_1 + iB_2)$$

$$X_1 = e^{\alpha t}e^{i\beta t}(B_1 + iB_2)$$

BY EULER

$$X_1 = e^{\alpha t}(\cos\beta t + i\sin\beta t)(B_1 + iB_2)$$

NOW SPLIT UP REAL & IMAGINARY

$$X_1 = e^{\alpha t}\left(\left[B_1\cos\beta t - B_2\sin\beta t\right] + i\left[B_1\sin\beta t + B_2\cos\beta t\right]\right)$$

SINCE $X_2 = \overline{X_1}$, IT IS THE SAME JUST FLIP THAT SIGN

$$X_2 = e^{\alpha t}\left(\left[\quad '' \quad\right] - i\left[\quad '' \quad\right]\right)$$

SINCE WE CAN ADD AN SCALE SOLUTIONS...

$$\frac{1}{2}(X_1 + X_2) = e^{\alpha t}\left[B_1\cos\beta t - B_2\sin\beta t\right]$$

$$\frac{1}{2i}(X_1 - X_2) = e^{\alpha t}\left[B_1\sin\beta t + B_2\cos\beta t\right]$$

BOTH SOLUTIONS

THEY ARE L.I. (WRONSKIAN) SO FORM A REAL GENERAL SOLUTION:

$$Y = c_1 e^{\alpha t}\left(B_1\cos\beta t - B_2\sin\beta t\right) + c_2 e^{\alpha t}\left(B_1\sin\beta t + B_2\cos\beta t\right)$$

SO LET'S APPLY THIS TO OUR EXAMPLE:

$\lambda_1 = 2+i$ so $\alpha = 2$, $\beta = 1$

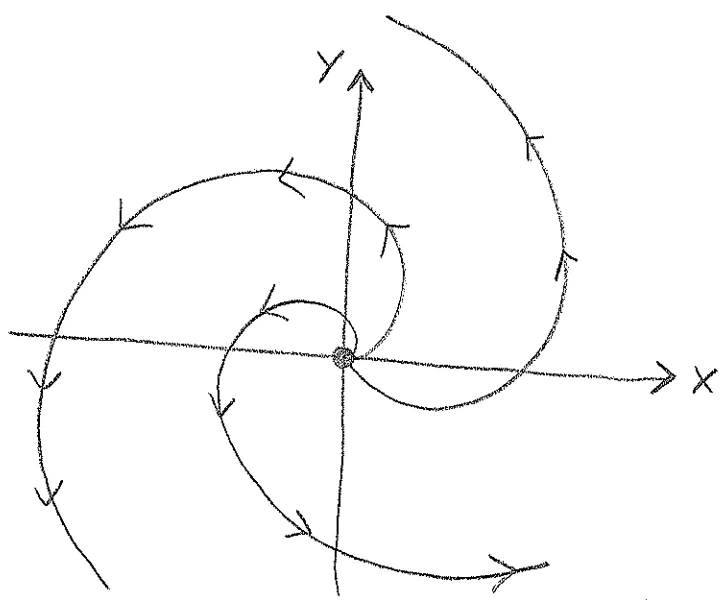
$V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$X = c_1 e^{2t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right) + c_2 e^{2t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right)$$

$$X = e^{2t}\left(c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}\right)$$

THE SOLUTION $e^{2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ IS $x = e^{2t}\cos t$, $y = e^{2t}\sin t$ (RECALL POLAR COORDS $x = r\cos\theta$, $y = r\sin\theta$)

LOOKS LIKE A SPIRAL SINCE THIS IS SIMILAR TO THE PARAMETRIZATION OF A CIRCLE BUT W/ GROWING RADIUS. THE OTHER TERM IS SIMILAR.




NOW OUR ONLY THING LEFT TO DISCUSS IS WHAT TO DO IF WE HAVE A REPEATED $\lambda$ WITHOUT ENOUGH L.I. EIGENVECTORS.
WHAT WE DO IS USE OTHER VECTORS CALLED GENERALIZED EIGENVECTORS TO COMPLETE OUR GENERAL SOLUTION.

THM SUPPOSE $A$ HAS AN EIGENVALUE $\lambda$ OF MULTIPLICITY TWO BUT WITH ONLY ONE L.I. EIGENVECTOR $V$ CORRESPONDING TO IT. THEN WE CAN FIND A VECTOR $P$ CALLED A GENERALIZED EIGENVECTOR s.t.

$$(A-\lambda I)P = V$$

THEN WE CAN USE THE SOLUTION

$$X_2 = te^{\lambda t}V + e^{\lambda t}P \quad \text{ALONG W/} \quad X_1 = e^{\lambda t}V$$

① $X_2$ WILL BE A SOLUTION

② $X_1$ & $X_2$ WILL BE L.I.

EX: FIND GEN. SOL. OF $X' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X$

$\lambda$'s: $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+1 = 0$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda-2)^2 = 0$$

$$\lambda = 2, 2$$

$V$'s: $(A-2I)V = 0$

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-x-y=0$

$-y=x$

LET $y=1$

$V = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ONLY ONE L.I. E.VECTOR

THUS WE FIND $P$ s.t.

$(A-2I)P = V$ → SAME SYSTEM OF EQ'S LEFT OF THE BAR AS ABOVE

$$\left(\begin{array}{cc|c} -1 & -1 & -1 \\ 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$-x-y=-1$

LET $y=1$, SO $x=0$

$P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

OUR GENERAL SOLUTION IS:

$$X = c_1 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2t}\left[t\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$ BY ABOVE