LAST TIME: WE TALKED ABOUT HOW TO SOLVE SYSTEMS OF DIFFY Q'S OF THE FORM:

$$Ax = x' \quad \text{(HOMOGENEOUS SYSTEM)}$$

① IF A HAS n L.I. EIGENVECTORS, THE GENERAL SOLUTION IS:

$$x = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + \dots + c_n e^{\lambda_n t} V_n \quad (V_i \text{ EIGENVECTOR W/ VALUE } \lambda_i)$$

② IF A HAS $< n$ L.I. EIGENVECTORS, WE FIND THE GENERALIZED EIGENVECTORS.

EX: A IS 2x2 W/ $\lambda_1 = \lambda_2 = 3$ AND ONLY ONE L.I. EIGENVECTOR V, THEN WE FIND P s.t.

$$(A - 3I)P = V$$

AND OUR GENERAL SOLUTION IS:

$$x = c_1 e^{3t} V + c_2 \underbrace{\left[t e^{3t} V + e^{3t} P \right]}_{\text{OUR SECOND L.I. SOLUTION}}$$

IF A (3x3) HAS A λ REPEATED 3 TIMES W/ ONLY ONE EIGENVECTOR V, WE FIND <u>ANOTHER</u> GENERALIZED EIGENVECTOR Q:

FIRST FIND V: $(A - \lambda I)V = 0$

THEN P: $(A - \lambda I)P = V$

THEN Q: $(A - \lambda I)Q = P$

SO IN GENERAL YOU FIND A GENERALIZED EIGENVECTOR FOR EACH EIGENVECTOR YOU ARE "MISSING." THUS IF λ OCCURS WITH MULTIPLICITY 7 AND HAS ONLY 4 L.I. EIGENVECTORS, YOU NEED TO FIND 3 GENERALIZED EIGENVECTORS.

AND OUR GENERAL SOLUTION WOULD BE:

$$x = c_1 e^{\lambda t} V + c_2 \left[t e^{\lambda t} V + e^{\lambda t} P \right] + c_3 \left[\tfrac{1}{2} t^2 e^{\lambda t} V + t e^{\lambda t} P + e^{\lambda t} Q \right]$$

AND SO ON.

RMK: THERE IS A THEOREM THAT SAYS THAT WE CAN <u>ALWAYS</u> FIND THESE GENERALIZED EIGENVECTORS TO GET OUR SOLUTION. THE PURPOSE OF THESE VECTORS IS THAT IF WE PUT THEM (AND THE EIGENVECTORS) AS THE COLUMNS OF A MATRIX AND CALL IT P, THEN THE MATRIX $J = P^{-1}AP$ IS ALMOST DIAGONAL WITH THE λ'S ON THE DIAGONAL, EX: $J = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ ← SO J IS DIAGONAL BUT WITH A FEW ONE'S ABOVE THE MAIN DIAG.

THIS IS CALLED THE <u>JORDAN FORM</u> AND IT IS FUNDAMENTAL TO LINEAR ALGEBRA OVER $\mathbb{C}$. (YOU DON'T NEED TO KNOW THIS...)

EX: FIND THE GENERAL SOLUTION: $X' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} X = AX$

FIND λ's: $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$ (DIAGONAL ELEMENTS)

FIND V's: FOR $\lambda = 1$ $\quad (A - I)V_1 = 0$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{array}\right)$$

RANK 2, SO 1 PARAM = 1 L.I. E.VECTOR

SO $-4x = 0$, $x = 0$

$\cancel{3x} + 6y + z = 0$

$z = -6y$

LET $y = 1$

$$\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} = V_1$$

THUS WE NEED TO FIND ONE GENERALIZED EIGENVECTOR CORRESPONDING TO $\lambda = 1$:

$$(A - I)P = V$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 \\ 3 & 6 & 1 & -6 \end{array}\right)$$

SO $-4x = 1$

$x = -\frac{1}{4}$

$3x + 6y + z = -6$

$-\frac{3}{4} + 6y + z = -6$

LET $y = \frac{1}{8}$ $\quad -\cancel{\frac{3}{4}} + \cancel{\frac{6}{8}} + z = -6$

$z = -6$

$$P = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{8} \\ -6 \end{pmatrix}$$

FOR $\lambda = 2$ $\quad (A - 2I)V_3 = 0$

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right)$$

SO $x = y = 0$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

AND OUR GENERAL SOLUTION IS:

$$X = c_1 e^t \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + c_2 \left[t e^t \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + e^t \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{8} \\ -6 \end{pmatrix} \right] + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

EX: FIND THE GENERAL SOLUTION: $X' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} X = AX$

FIND λ's: $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2+4\right] = 0$

SO $\lambda = 1$ OR $(1-\lambda)^2 = -4$

$1-\lambda = \pm 2i$

$1 \pm 2i = \lambda$

$\lambda_1 = 1,\ \lambda_2 = 1+2i,\ \lambda_3 = \overline{\lambda}_2$

FIND V's: $(A-I)V_1 = 0$

$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right)$

$2x - 2z = 0,\ x = z$

$3x + 2y = 0$

$y = -\frac{3}{2}x$

LET $x = 2$: $V_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$(A-(1+2i)I)V_2 = 0$

$\left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right)$

E_1: $-2ix = 0$, $x = 0$

E_2: $\cancel{2x} - 2iy - 2z = 0$

$-iy = z$

E_3: $\cancel{3x} + 2y - 2iz = 0$

$y = iz$

$-iy = z$

SO LET $y = 1$:

$V_2 = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$

$V_3 = \overline{V}_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$

OUR COMPLEX GENERAL SOLUTION IS THEN:

$$X = c_1 e^{t}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 e^{(1+2i)t}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} + c_2 e^{(1-2i)t}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

TO FIND THE REAL SOLUTION WE CAN USE OUR FORMULA ON THE BOTTOM OF PAGE 105 W/ $\alpha = 1$, $\beta = 2$, $B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ SINCE $\lambda_1 = 1+2i$, $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

CAREFUL TO NOT MIX UP AND USE λ_1 TO FIND α & β THEN V_2 TO FIND B_1 & B_2! YOU WILL BE OFF BY A NEGATIVE SIGN UNLESS YOU ARE CONSISTENT AND USE V_1, λ_1 OR V_2, λ_2 TO GET THESE α, β, & B's.

$$\boxed{X = c_1 e^{t}\begin{pmatrix}2\\-3\\2\end{pmatrix} + c_2 e^{t}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\cos 2t - \begin{pmatrix}0\\0\\-1\end{pmatrix}\sin 2t\right) + c_3 e^{t}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\sin 2t + \begin{pmatrix}0\\0\\-1\end{pmatrix}\cos 2t\right)}$$

IF YOU DON'T LIKE PLUGGING INTO FORMULAS RECALL HOW WE GOT THE SOL. ON 105; WE TOOK A COMPLEX SOLUTION $e^{(\alpha+i\beta)t}(B_1 + iB_2) = \left[\text{REAL STUFF } 1\right] + i\left[\text{REAL STUFF } 2\right]$

AND SPLIT IT UP INTO STUFF W/ i'S AND STUFF W/O i'S. THESE WERE OUR REAL SOLUTIONS.

SO IN OUR CASE:

$$e^{(1+2i)t}\begin{pmatrix}0\\1\\-i\end{pmatrix} = e^{t}e^{2it}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} - i\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$= e^{2t}\left(\cos 2t + i\sin 2t\right)\left[\begin{pmatrix}0\\1\\0\end{pmatrix} - i\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$= \underbrace{e^{2t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\cos 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}\sin 2t\right]} + i\,\underbrace{e^{2t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\sin 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}\cos 2t\right]}$$

OUR TWO REAL SOLUTIONS ABOVE (THE SECOND ONE WE DROP THE i)

ONE LAST EXAMPLE:

EX: SUPPOSE THE MOTION OF A PARTICLE IS DESCRIBED BY THE SYSTEM OF DIFFY Q'S:

$$X' = \begin{pmatrix}1&0&0\\2&1&-2\\3&2&1\end{pmatrix}X, \quad X(0) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$$

FIND THE EQUATION OF MOTION.

THE GENERAL SOL. IS ABOVE SO WE USE TO FIND $c_1, c_2, \& c_3$:

$$X(0) = \begin{pmatrix}2\\4\\-1\end{pmatrix} = c_1\begin{pmatrix}2\\-3\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

SO $2c_1 = 2$, $c_1 = 1$

AND $-3c_1 + c_2 = 4$, $c_2 = 7$

AND $2c_1 - c_3 = -1$, $c_3 = 3$ PLUGGING THESE IN & SIMPLIFYING:

$$\boxed{X = e^{t}\begin{pmatrix}2\\-3\\2\end{pmatrix} + e^{2t}\left[\begin{pmatrix}0\\7\cos t\\7\sin t\end{pmatrix} + \begin{pmatrix}0\\3\sin t\\-3\cos t\end{pmatrix}\right]}$$

POWER SERIES SOLUTIONS TO DIFFY Q'S

NOW WHAT WE WILL DO IS A SORT OF "METHOD OF AN INFINITE NUMBER OF UNDETERMINED COEFFICIENTS" TO SOLVE DIFFY Q'S. NAMELY, WE ASSUME THERE IS A SOLUTION OF THE FORM $y = c_0 + c_1 x + c_2 x^2 + \dots = \sum_{n \geq 0} c_n x^n$ AND SOLVE FOR THE C'S.

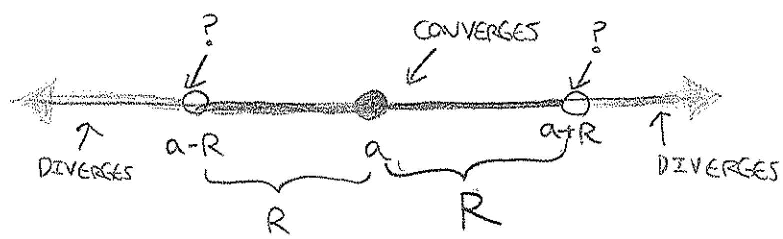
FIRST A REVIEW OF POWER SERIES:

DEF: A POWER SERIES CENTERED AT a IS AN INFINITE SERIES OF THE FORM:

$$\sum_{n \geq 0} c_n (x-a)^n$$

SINCE THERE ARE AN INFINITE # OF TERMS THIS SUM MAY DIVERGE FOR CERTAIN VALUES OF X. THE VALUES OF X FOR WHICH THE SUM CONVERGES ARE SAID TO BE IN THE <u>RADIUS OF CONVERGENCE</u>, USUALLY DENOTED R.

THM: THE VALUES OF X FOR WHICH $\sum_{n \geq 0} c_n (x-a)^n$ IS CONVERGANT IS AN INTERVAL OF THE FORM $(a-R, a+R)$ WHERE R IS THE RADIUS OF CONVERGENCE. THE SERIES DIVERGES FOR $x < a-R$, $x > a+R$. FOR $x = a \pm R$, EITHER COULD HAPPEN.



SO WHAT IS THE UTILITY OF POWER SERIES? FIRSTLY,

1. MOST FUNCTIONS f WITH ALL DERIVATIVES f', f'', f''' ETC. EXISTING HAVE A POWER SERIES CONVERGING TO IT NEAR ANY POINT. (i.e. IT <u>IS</u> THE POWER SERIES LOCALLY)

2. POWER SERIES ARE VERY EASY TO INTEGRATE AND DIFFERENTIATE. JUST DO IT TERM BY TERM:

$$\left(\sum_{n \geq 0} c_n x^n\right)' = \sum_{n \geq 0} n c_n x^{n-1}$$

$$\int \left(\sum_{n \geq 0} c_n x^n\right) = \sum_{n \geq 0} \frac{c_n}{n+1} x^{n+1} + C$$

THESE ARE <u>NONTRIVIAL FACTS</u> FROM ANALYSIS.

FUNCTIONS SATISFYING (1) ARE CALLED ANALYTIC. SOME EXAMPLES OF ANALYTIC FUNCTIONS ARE POLYNOMIALS, $\sin x$, $\cos x$, e^x, $\ln x$. SO OUR ASSUMPTION THAT OUR SOLUTIONS TO SOME DIFFY Q ARE A POWER SERIES (I.E. AN ANALYTIC FUNCTION) ARE NOT REALLY AS RESTRICTIVE AS ONE MAY HAVE THOUGHT.

SHIFTING VARIABLES

WE WILL OFTEN BE CHANGING OUR COUNTING VARIABLES IN OUR POWER SERIES. THE REASON IS TO BE ABLE TO COMBINE MULTIPLE POWER SERIES INTO ONE.

EX: COMBINE $\sum_{n\geq 0}(n+1)C_{n+2}X^{n+2} + \sum_{n\geq 1} b_{n-1}X^{n-1}$

(1) CHANGE VARIABLES TO GET EACH POWER SERIES INTO THE FORM $\sum_{k\geq ?}(\text{STUFF})X^k$

SO IN THE FIRST SUM LET $k=n+2$ (SO: $k-1=n+1$) IN THE SECOND SUM LET $k=n-1$

$$\sum_{k\geq 2}(k-1)C_k X^k + \sum_{k\geq 0} b_k X^k$$

SINCE n STARTS AT 0 & $k=n+2$

SINCE n STARTS AT 1 AND $k=n-1$

(2) NOW PULL OUT TERMS UNTIL BOTH SERIES START AT THE SAME POWER OF X

$$\sum_{k\geq 2}(k-1)C_kX^k + \sum_{k\geq 0} b_kX^k = b_0+b_1x+\sum_{k\geq 2}(k-1)C_kX^k+\sum_{k\geq 2}b_kX^k$$

PULL OUT $k=0,1$ TERMS

(3) COMBINE!

$$\boxed{b_0+b_1x+\sum_{k\geq 2}\left[(k-1)C_k+b_k\right]X^k}$$

WE WILL DO THIS A LOT.

2ND ORDER DIFFY Q'S (HOMOGENEOUS)

WE WILL NOW DISCUSS SOLVING 2ND ORDER DIFFY Q'S (OF THE FORM:)

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0$$

WE DIVIDE BY $a_2(x)$ TO GET IT INTO STANDARD FORM

$$y'' + P(x)y' + Q(x)y = 0$$

THM: IF BOTH P AND Q (AS ABOVE) ARE ANALYTIC AT A POINT $x=a$, THEN THERE EXISTS TWO L.I. POWER SERIES SOLUTIONS OF THE FORM

$$y = \sum_{n\geq 0} c_n (x-a)^n$$

NOW WE NEED TO NOW WHEN THE FUNCTIONS P & Q ARE ANALYTIC. IN MOST EXAMPLES WE WILL DO, P AND Q WILL BE RATIONAL FUNCTIONS WHICH IS ANOTHER WAY TO SAY THEY ARE A FRACTION WITH A POLYNOMIAL NUMERATOR & DENOMINATOR.

FACT: A RATIONAL FUNCTION IS ANALYTIC EVERYWHERE EXCEPT AT THE ZEROS IN ITS DENOMINATOR.

EX: $P(x) = \frac{3x+4}{x^2-1}$ IS ANALYTIC EVERYWHERE BUT $x = \pm 1$

$P(x) = \frac{1}{x(x-3)}$ ANALYTIC OUTSIDE OF $x=0,3$

DEF: WHEN SOLVING $y'' + P(x)y' + Q(x)y = 0$

AN ORDINARY POINT IS A VALUE OF x WHERE P & Q ARE BOTH ANALYTIC

OTHERWISE IT IS CALLED A SINGULAR POINT.

LET'S DO SOME EXAMPLES:

FIND THE GENERAL SOLUTION OF: $y'' + xy = 0$ (USE SERIES CENTERED AT $x=0$)

HERE $P(x) = 0$, $Q(x) = x$ ARE BOTH ANALYTIC. THUS WE CAN FIND 2 L.I. SOL'S OF THE FORM: $y = \sum_{n\geq 0} c_n x^n$

$$y' = \sum_{n\geq 0} n c_n x^{n-1} = \sum_{n\geq 1} n c_n x^{n-1} \quad (n=0 \text{ TERM IS } 0)$$

$$y'' = \sum_{n\geq 1} n(n-1) c_n x^{n-2} = \sum_{n\geq 2} n(n-1) c_n x^{n-2} \quad (n=1 \text{ TERM IS } 0)$$

PLUGGING THIS BACK IN:

MULT IN

$$\sum_{n\geq 2} n(n-1)c_n x^{n-2} + x\sum_{n\geq 0} c_n x^n = 0$$

$$\underbrace{\sum_{n\geq 2} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n\geq 0} c_n x^{n+1}}_{k=n+1} = 0$$ NOW WE DO OUR STEPS TO COMBINE THEM:

$$\sum_{k\geq 0}(k+2)(k+1)c_{k+2}x^k + \sum_{k\geq 1} c_{k-1}x^k = 0$$

PULL OUT $k=0$ TERM

$$2c_2 + \underbrace{\sum_{k\geq 1}(k+2)(k+1)c_{k+2}x^k + \sum_{k\geq 1} c_{k-1}x^k}_{\text{COMBINE!}} = 0$$

$$2c_2 + \sum_{k\geq 1}\left[(k+2)(k+1)c_{k+2} + c_{k-1}\right]x^k = 0$$

NOW THE ONLY WAY THIS FUNCTION IS ZERO FOR <u>ALL X</u> IS IF EVERY COEFFICIENT IN THE POWER SERIES IS 0. IN OTHER WORDS:

$2c_2 = 0$ $\qquad (k+2)(k+1)c_{k+2} + c_{k-1} = 0$

$c_2 = 0$

$$c_{k+2} = \frac{-c_{k-1}}{(k+2)(k+1)}$$ HOLDS FOR $k\geq 1$ (I.E. THE K'S IN OUR SUM ABOVE)

THIS IS CALLED THE <u>RECURRENCE RELATION</u>

SO WE HAVE:

$k=1$: $\quad c_3 = \dfrac{-c_0}{2\cdot 3}$

$k=2$: $\quad c_4 = \dfrac{-c_1}{3\cdot 4}$

$k=3$: $\quad c_5 = \dfrac{-c_2}{4\cdot 5} = 0 \quad (c_2=0)$

$k=4$: $\quad c_6 = \dfrac{-c_3}{5\cdot 6} = \dfrac{c_0}{2\cdot 3\cdot 5\cdot 6}$ BY ABOVE

$k=5$: $\quad c_7 = \dfrac{-c_4}{6\cdot 7} = \dfrac{c_1}{3\cdot 4\cdot 6\cdot 7}$ BY ABOVE

$k=6$: $\quad c_8 = \dfrac{-c_5}{7\cdot 8} = 0 \quad (c_5=0)$

ETC.

SO WE GET THAT FOR <u>ANY</u> C_0 AND C_1:

$$Y = C_0 + C_1 X + 0 - \frac{C_0}{2\cdot 3} X^3 - \frac{C_1}{3\cdot 4} X^4 + 0 + \frac{C_0}{2\cdot 3\cdot 5\cdot 6} X^6 + \frac{C_1}{3\cdot 4\cdot 6\cdot 7} X^7 + 0 - \dots$$

IS A SOLUTION.

$$= C_0 \Big(\underbrace{1 - \frac{1}{2\cdot 3} X^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6} X^5 - \dots}_{Y_1} \Big) + C_1 \Big(\underbrace{X - \frac{1}{3\cdot 4} X^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7} X^7 - \dots}_{Y_2} \Big)$$

Y_1 AND Y_2 WILL BE OUR TWO L.I. SOLUTIONS CORRESPONDING TO $C_1 = 1, C_2 = 0$ & $C_1 = 0, C_2 = 1$

HERE $\displaystyle Y_1 = 1 + \sum_{k\geq 1} \frac{(-1)^k}{2\cdot 3 \dots (3k-1)(3k)} X^{3k}$

$$Y_2 = X + \sum_{k\geq 1} \frac{(-1)^k}{3\cdot 4 \dots (3k)(3k+1)} X^{3k+1}$$

EX: FIND THE GENERAL SOLUTION: $(X^2+1)Y'' + XY' - Y = 0$ (CENTER SOL'S AT $X=0$)

$P(X) = \dfrac{X}{X^2+1}$, $Q(X) = \dfrac{-1}{X^2+1}$ ARE BOTH ANALYTIC AT $X=0$. SO WE CAN FIND

2 L.I. SOL'S OF THE FORM: $\displaystyle Y = \sum_{n\geq 0} C_n X^n, \quad Y' = \sum_{n\geq 1} n C_n X^{n-1}, \quad Y'' = \sum_{n\geq 2} n(n-1) C_n X^{n-2}$

PLUG THESE IN: $\displaystyle (X^2+1)\sum_{n\geq 2} n(n-1)C_n X^{n-2} + X \sum_{n\geq 1} n C_n X^{n-1} - \sum_{n\geq 0} C_n X^n = 0$

$$\underbrace{\sum_{n\geq 2} n(n-1)C_n X^n}_{k=n} + \underbrace{\sum_{n\geq 2} n(n-1) C_n X^{n-2}}_{k=n-2} + \underbrace{\sum_{n\geq 1} n C_n X^n}_{k=n} - \underbrace{\sum_{n\geq 0} C_n X^n}_{k=n} = 0$$

$$\sum_{k\geq 2} k(k-1)C_k X^k + \sum_{k\geq 0}(k+2)(k+1)C_{k+2} X^k + \sum_{k\geq 1} k C_k X^k - \sum_{k\geq 0} C_k X^k = 0$$

PULL OUT $k=0,1$ TERMS   PULL OUT $k=1$ PULL OUT $k=0,1$

$$2C_2 + 6C_3 X + \cancel{C_1 X} - (C_0 + \cancel{C_1 X}) + \sum_{k\geq 2}\Big[k(k-1)C_k + (k+2)(k+1)C_{k+2} + kC_k - C_k\Big] X^k$$

$$(2C_2 - C_0) + 6C_3 X + \sum_{k\geq 2}\big[\text{ SAME }\big] X^k$$

AGAIN EACH COEFF. MUST BE ZERO. THIS IMPLIES:

$2C_2 - C_0 = 0$ $\qquad$ $6C_3 = 0$ $\qquad$ $\left[k(k-1)+(k-1)\right]C_k + (k+2)(k+1)C_{k+2} = 0$ FOR $k \geq 2$

$C_2 = \frac{1}{2}C_0$ $\qquad$ $C_3 = 0$

$$\cancel{(k+1)}(k-1)C_k + (k+2)\cancel{(k+1)}C_{k+2} = 0$$

$$\boxed{C_{k+2} = \frac{1-k}{k+2}C_k}$$ RECURRENCE RELATION

$k=2:\quad C_4 = \frac{-1}{4}C_2 = -\frac{1}{8}C_0$

$k=3:\quad C_5 = \frac{-2}{5}C_3 = 0$

$k=4:\quad C_6 = \frac{-3}{6}C_4 = \frac{1}{16}C_0$

$k=5:\quad C_7 = \frac{-4}{7}C_5 = -\frac{8}{35}C_3 = 0$

$k=6:\quad C_8 = \frac{-5}{8}C_6 = \frac{-5}{8\cdot 16}C_0$ ETC.

SO OUR GENERAL SOLUTION IS:

$$Y = C_0 + C_1X + \frac{1}{2}C_0X^2 + 0 - \frac{1}{8}C_0X^4 + 0 + \frac{1}{16}C_0X^6 + 0 - \frac{5}{8\cdot 16}C_0X^8 + \ldots$$

$$= C_0\Big(\underbrace{1 - \frac{1}{2}X^2 - \frac{1}{8}X^4 + \frac{1}{16}X^6 - \frac{5}{8\cdot 16}X^8 + \ldots}_{Y_1}\Big) + C_1\underset{Y_2 = X}{\underset{\uparrow}{X}}$$

THIS EXAMPLE SHOWS HOW ONE SOLUTION COULD ONLY HAVE FINITELY MANY TERMS.

RADIUS OF CONVERGENCE

HOW DO WE FIND OUT WHERE OUR SOLUTIONS CONVERGE? (RECALL THEY NEED NOT CONVERGE ON ALL OF $\mathbb{R}$)

METHOD 1 IF YOU CAN WRITE Y IN THE FORM $Y = \sum_{n\geq 0} C_nX^n$ YOU CAN APPLY THE RATIO TEST

RATIO TEST: TO SEE IF $\sum a_n$ CONVERGES, LOOK AT:

$$\text{IF} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \begin{cases} < 1, & \text{CONVERGES} \\ > 1, & \text{DIVERGES} \\ = 1, & \text{CAN'T TELL (USE ANOTHER TEST)} \end{cases}$$

EX: FIND THE RADIUS OF CONVERGENCE OF $y = \sum_{n \geq 0} \frac{2^n}{n} x^n$

APPLY RATIO TEST $\left(a_n = \frac{2^n}{n} x^n \right)$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{\frac{2^{n+1}}{n+1} x^{n+1}}{\frac{2^n}{n} x^n} \right| = \lim \left| \frac{n}{n+1} 2x \right| = 2|x| \overbrace{\lim \left| \frac{n}{n+1} \right|}^{=1} = 2|x| < 1$$

IFF $|x| < \frac{1}{2}$

SO THE SERIES CONVERGES IF $|x| < \frac{1}{2}$

DIVERGES IF $|x| > \frac{1}{2}$

AND THE RADIUS OF CONVERGENCE IS $R = \frac{1}{2}$

THERE IS A NICE THEOREM WE WILL USE TO FIND THE RADIUS OF CONVERGENCE OF A SERIES SOLUTION:

THM: IF y IS A SERIES SOLUTION TO $y'' + P(x)y' + Q(x)y = 0$, CENTERED AT $x = a$ (i.e. $y = \sum c_n (x-a)^n$) THEN THE RADIUS OF CONVERGENCE OF y IS JUST THE DISTANCE FROM a TO THE NEAREST SINGULAR POINT.

EX: LET'S FIND THE RADII OF CONVERGENCE IN OUR TWO EXAMPLES WE DID:

FIRST WE SOLVED: $y'' + xy = 0$ WITH A SERIES CENTERED AT 0.

THIS HAS NO SINGULAR POINTS SO $R = \infty$

THEN WE SOLVED: $(x^2+1)y'' + xy' - y = 0$

$$y'' + \frac{x}{x^2+1} y' - \frac{1}{x^2+1} y = 0$$

AGAIN CENTERED AT $x=0$

THIS IS SINGULAR WHEN $x^2+1 = 0$, OR $x = \pm i$

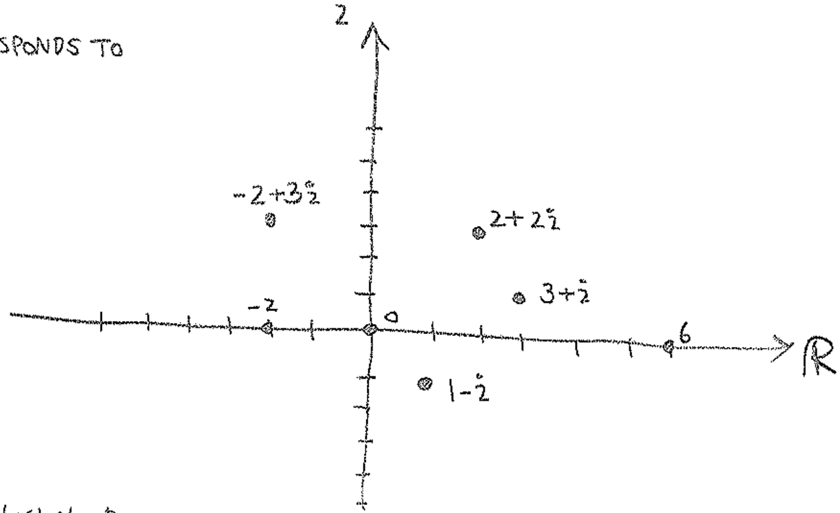
THE DISTANCE FROM $x=0$ TO $x = \pm i$ IN THE COMPLEX PLANE IS 1 SO $R = 1$.

THE COMPLEX PLANE IS JUST $\mathbb{R}^2$ BUT WE PLOT COMPLEX #'S ON IT

(SO THE y-AXIS IS THE i-AXIS, x-AXIS IS THE $\mathbb{R}$-AXIS)

So (x,y) CORRESPONDS TO

$x+iy$:



SO WE USE THE USUAL DISTANCE IN $\mathbb{R}^2$ TO FIND THE DISTANCE BETWEEN COMPLEX #'s.

EX: FIND THE RADIUS OF CONVERGENCE OF THE SOLUTIONS CENTERED AT $x=0$ TO:

$$(x^2-2x+4)y'' + xy' + (x+2)y = 0$$

$$y'' + \frac{x}{(x^2-2x+4)}y' + \frac{x+2}{(x^2-2x+4)}y = 0$$

SINGULAR PTS WHEN $x^2-2x+4=0$

$$x = \frac{1}{2}\left(2 \pm \sqrt{4-16}\right)$$

$$x = 1 \pm \sqrt{3}\, i$$

HERE $R=2$ SINCE THE DISTANCE FROM THE ORIGIN TO $(1,\sqrt{3})$ IS $\sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$