Quiz 2

NAME:______________________________________

For your own benefit show all of your work, keep it neat, and if you write more than one solution make clear which is your answer or I will just grade whichever one is closest to the top of the paper.

1. (10pts) Find the general solution to the system of equations:

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} X$$

First we find the eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = -\lambda(4-\lambda) + 4 = (\lambda - 2)^2 = 0$$

Thus we have $\lambda_1 = \lambda_2 = 2$. Now we find the eigenvectors:

$$(A - 2I)v = 0 \mapsto \left(\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array} \right) \mapsto \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Thus our only condition is $-2x + y = 0$ so $y = 2x$ and we can let $v = \binom{1}{2}$. Since this is our only linearly independent eigenvector (we only have 1 parameter) we need to find the generalized eigenvector P:

$$(A - 2I)P = v \mapsto \left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array} \right) \mapsto \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

So our only condition is that $-2x + y = 1$, so let $x = 1$ and get $P = \binom{1}{3}$. Thus our general solution is:

$$X = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[te^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]$$

Note that you may have picked different v's and P's, but will still have the correct general solution. Now the initial conditions will get us:

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solving this system should get us $c_1 = 2$ and $c_2 = -1$. So our final answer is:

$$X = e^{2t} \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \left[te^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - te^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

2. (10pts) Find the recurrence relation corresponding to a series solution to the following differential equation (centered at 0):

$$(x^2+1)y'' + xy' + 3y = 0$$

We recognize that $x = 0$ is not a singular point of our equation and thus we can find two power series solutions. So we will plug in the following into the above equation:

$$y = \sum_{n\geq 0} c_n x^n \qquad y' = \sum_{n\geq 1} nc_n x^{n-1} \qquad y'' = \sum_{n\geq 2} n(n-1)c_n x^{n-2}$$

So we get:

$$(x^2+1)\sum_{n\geq 2} n(n-1)c_n x^{n-2} + x\sum_{n\geq 1} nc_n x^{n-1} + 3\sum_{n\geq 0} c_n x^n = 0$$

$$\underbrace{\sum_{n\geq 2} n(n-1)c_n x^n}_{\text{let k = n}} + \underbrace{\sum_{n\geq 2} n(n-1)c_n x^{n-2}}_{\text{let k = n - 2}} + \underbrace{\sum_{n\geq 1} nc_n x^n}_{\text{let k = n}} + \underbrace{\sum_{n\geq 0} 3c_n x^n}_{\text{let k = n}} = 0$$

$$\sum_{k\geq 2} k(k-1)c_k x^k + \underbrace{\sum_{k\geq 0}(k+2)(k+1)c_{k+2}x^k}_{\substack{\text{pull out}\\ \text{k=0,1 terms}}} + \underbrace{\sum_{k\geq 1} kc_k x^k}_{\substack{\text{pull out}\\ \text{k=1}\\ \text{term}}} + \underbrace{\sum_{k\geq 0} 3c_k x^k}_{\substack{\text{pull out}\\ \text{k=0,1}\\ \text{terms}}} = 0$$

$$(2c_2+3c_0)+(6c_3+c_1+3c_1)x+\sum_{k\geq 2}\left[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k + 3c_k\right]x^k = 0$$

So our recurrence relation comes from setting the bracketed term equal to zero:

$$[k(k-1)+k+3]c_k + (k+2)(k+1)c_{k+2} = 0$$
$$[k^2+3]c_k + (k+2)(k+1)c_{k+2} = 0$$
$$c_{k+2} = \frac{-[k^2+3]c_k}{(k+1)(k+2)}$$

This recurrence holds for $k \geq 2$, (actually it holds for $k \geq 0$ if we check.)

Formulas

Complex solutions:
$\lambda = \alpha + i\beta$, $v = B_1 + iB_2$
$X = c_1 e^{\alpha t}\left[B_1 \cos \beta t - B_2 \sin \beta t\right] + c_2 e^{\alpha t}\left[B_1 \sin \beta t + B_2 \cos \beta t\right]$
Repeated eigenvalues:
$X = c_1 e^{\lambda t} v + c_2 [t e^{\lambda t} v + e^{\lambda t} P]$