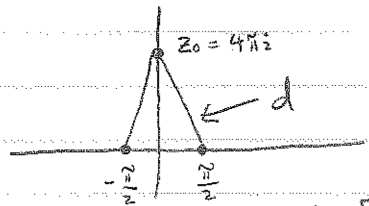
I) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ HAS POLES AT $z = \frac{\pi}{2} + \pi n$



$$d = \sqrt{\left(\tfrac{\pi}{2}\right)^2 + (4\pi)^2} = \frac{\pi}{2}\sqrt{65} \quad \text{(B)}$$

II) $f(z) = \dfrac{7z-3}{z(z-1)^2}$ on $0 < |z-1| < 1$

$b_1 + b_2 + b_3 + a_0 + a_1 + a_2 = ?$

$f(z) = (z-1)^{-2}\left[7 - \frac{3}{z}\right]$

$$\frac{3}{z} = \frac{3}{z-1+1} = \frac{3}{1-(-[z-1])} = 3\sum_{n\geq 0}(-1)^n (z-1)^n$$

$$f(z) = \frac{7}{(z-1)^2} - 3\sum_{n\geq 0}(-1)^n (z-1)^{n-2}$$

$$= \frac{4}{(z-1)^2} + \frac{3}{z-1} - 3 + 3(z-1) - 3(z-1)^2 + \dots$$

SUM = 4 (E)

III) $f(z) = 2z^2 - 2\sin(z^2)$

$$= 2z^2 - 2\sum_{n\geq 0}(-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!}$$

$$= 2z^2 - 2\left[z^2 - \frac{z^6}{3!} + \frac{z^9}{5!} - \dots\right]$$

$$= \frac{2}{3!}z^6 - \frac{2}{5!}z^9 + \dots$$

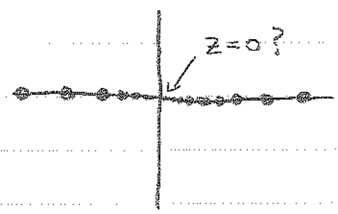
↑ ORDER 6 (C)

(IV) $\tan\left(\frac{1}{z}\right) = \dfrac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$ HAS A POLE AT $\frac{1}{z} = \frac{\pi}{2} + n\pi$

$$z = \frac{1}{\frac{\pi}{2} + n\pi}$$

$$z = \frac{2}{\pi + 2\pi n}$$

NOTICE THAT THESE POLES APPROACH $z=0$ AS $n \to \infty$. SO NOW WE WANT TO INVESTIGATE $f = \tan\left(\frac{1}{z}\right)$ AT $z=0$:



IF f HAD A POLE OF ORDER p AT $z=0$ THEN $z^p f(z)$ IS HOLOMORPHIC AT p AND SO:

$$z^p f(z) = \sum_{n \geq 0} a_n z^n \quad \text{ON SOME DISC } |z| < \varepsilon$$

THUS IF $z \neq 0$, WE HAVE

$$f(z) = \sum_{n \geq 0} a_n z^{n-p} \quad \text{ON } 0 < |z| < \varepsilon$$

I.E. f IS ANALYTIC ON THE PUNCTURED DISC, WHICH CANNOT HAPPEN SINCE f HAS POLES APPROACHING $z=0$. THUS f CANNOT HAVE A POLE AT $z=0$. THE SAME ARGUMENT WORKS FOR A REMOVABLE SINGULARITY. THUS, $\tan\left(\frac{1}{z}\right)$ HAS AN ESSENTIAL SINGULARITY AT $z=0$, AND THUS SO DOES $\dfrac{\tan\left(\frac{1}{z}\right)}{z^2 - z}$.

A

WHY WAS THIS NECESSARY? EVEN THOUGH $\sin\left(\frac{1}{z}\right)$ AND $\cos\left(\frac{1}{z}\right)$ HAVE AN ESSENTIAL SINGULARITY AT $z=0$ IT DOES NOT MEAN THAT SO DOES THEIR RATIO. A TRIVIAL COUNTEREXAMPLE WOULD BE

$f(z) = 1 = \dfrac{\sin\left(\frac{1}{z}\right)}{\sin\left(\frac{1}{z}\right)}$ HAS NO SINGULARITY AT $z=0$.

THE ESSENTIAL POINT OF THIS PROBLEM IS THAT POLES ARE <u>ISOLATED</u>, MEANING THAT IF f HAS A POLE AT z_0 THEN THERE IS SOME ε-BALL $|z - z_0| < \varepsilon$ SUCH THAT z_0 IS f's <u>ONLY</u> SINGULARITY INSIDE $|z - z_0| < \varepsilon$.

(V) $\gamma : |z| = \frac{3}{2}$ ↺

$$\oint_\gamma \frac{e^{2z}}{(z^3+2z^2)}\,dz = \oint f\,dz$$

POLES: $z=0$ ORD 2, $z=-2$ ORD 1

$$= 2\pi i \,\mathrm{Res}(f,0)$$
$$= 2\pi i \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right]$$
$$= 2\pi i \left.\frac{d}{dz}\left(\frac{e^{2z}}{z+2}\right)\right|_{z=0}$$
$$= 2\pi i \left.\frac{2e^{2z}(z+2) - e^{2z}}{(z+2)^2}\right|_{z=0} = 2\pi i\left(\frac{1}{4}\right)(4-1) = \frac{3\pi i}{2}$$

(E)

(VI) $f(x) = 0$ on $[-\pi, 0]$
$= x$ on $[0,\pi]$

$c_{-2} + c_{-1} + c_0 = ?$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad u = x \qquad dv = e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\int_0^{\pi} xe^{-inx}\,dx \qquad du = dx \qquad v = -\frac{1}{in}e^{-inx}$$
$$= \frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx} + \frac{1}{in}\int e^{-inx}\,dx\right]$$
$$= \frac{1}{2\pi}\left[-\frac{1}{in}xe^{-inx} - \frac{1}{(in)^2}e^{-inx}\right]\Big|_0^{\pi}$$
$$= \frac{1}{2\pi}\left[\left(-\frac{x}{in} + \frac{1}{n^2}\right)e^{-inx}\right]\Big|_0^{\pi}$$
$$= \frac{1}{2\pi}\left[\left(-\frac{\pi}{in} + \frac{1}{n^2}\right)(-1)^n - \left(0 + \frac{1}{n^2}\right)\right]$$
$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n^2} + \frac{\pi i}{n}\right) - \frac{1}{n^2}\right]$$
$$c_{-2} = \frac{1}{2\pi}\left[\frac{1}{4} - \frac{\pi i}{2} - \frac{1}{4}\right] = -\frac{i}{4}$$
$$c_{-1} = \frac{1}{2\pi}\left[(-1)(1 - \pi i) - 1\right]$$
$$= \frac{1}{2\pi}\left[-2 + \pi i\right]$$
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int x\,dx$$
$$= \frac{1}{4\pi}x^2\Big|_0^{\pi}$$
$$= \frac{\pi}{4}$$

$$c_0 + c_{-1} + c_{-2} = \frac{\pi^2 - 4}{4\pi} + \frac{i}{4}$$

(D)

VII $f(x) = -1 \quad -\pi < x < 0, \quad f = 0 \quad 0 < x < \pi$

$a_0 + a_1 + a_2 + a_3 + b_0 + b_1 + b_2 + b_3 = ?$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\, dx = \frac{1}{\pi} \int_{-\pi}^{0} -1\, dx = \frac{1}{\pi}(-x)\Big|_{-\pi}^{0} = \frac{1}{\pi}\left(0 + (-\pi)\right) = -1$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos(nx)\, dx$

$= \frac{-1}{\pi} \int_{-\pi}^{0} \cos nx\, dx$

$= \frac{-1}{n\pi} \left[\sin nx\right]\Big|_{-\pi}^{0}$

$= 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin(nx)\, dx$

$= \frac{-1}{\pi} \int_{-\pi}^{0} \sin(nx)\, dx$

$= \frac{1}{n\pi} \left[\cos(nx)\right]\Big|_{-\pi}^{0}$

$= \frac{1}{n\pi} \left[1 - \cos(n\pi)\right]$

$= \frac{1}{n\pi} \left[1 - (-1)^n\right]$

$b_1 = \frac{1}{\pi}[2] = \frac{2}{\pi}$

$b_2 = 0$ ANS: $\frac{2}{\pi} + \frac{2}{3\pi} - 1 = \frac{8 - 3\pi}{3\pi}$ E

$b_3 = \frac{1}{3\pi}[2] = \frac{2}{3\pi}$

VIII $\frac{1}{2}x^2 y'' + x y' + (\lambda x^2 - 1) y = 0$

$y'' + \frac{2}{x} y' + \left(2\lambda - \frac{2}{x^2}\right) y = 0$

$\mu = e^\wedge\left(\int \frac{2}{x}\, dx\right) = e^{2\ln x} = e^{\ln x^2} = x^2$

$x^2 y'' + 2x y' + (2\lambda x^2 - 2) y = 0$

$\left[x^2 y'\right]' + (2\lambda x^2 - 2) y = 0$

SELF-ADJOINT FORM

WEIGHT FUNCTION = $2x^2$ (OR JUST x^2 WILL DO) C

IX $\quad y'' - 2xy' + 2ny = 0 \quad$ ON $[0, \infty)$

HAS NONZERO SOLS EXACTLY WHEN $n \in \mathbb{Z}_{\geq 0}$ (I.E. THE EIGENVALUES)

$$a(x) = e^{\wedge}\left(\int -2x\,dx\right) = e^{-x^2}$$

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2n e^{-x^2} y = 0$$

$$[e^{-x^2} y']' + 2n e^{-x^2} y = 0$$

SELF-ADJOINT FORM

WEIGHT FUNCTION $= 2e^{-x^2}$ (OR JUST e^{-x^2})

SO SOLS SATISFY:

$$\int_0^\infty e^{-x^2} H_n(x)\, H_m(x)\, dx = 0$$

X $\quad \displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} \qquad z = e^{i\theta} \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$dz = iz\,d\theta$$

$$= \int \frac{1}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i} \int \frac{1}{\frac{1}{2}z^2 + 2z + \frac{1}{2}}\, dz$$

$$= \frac{2}{i} \int \frac{1}{z^2 + 4z + 1}\, dz$$

$$z^2 + 4z + 1 = 0$$

$$z = \frac{1}{2}\left(-4 \pm \sqrt{16 - 4}\right)$$

$$= \frac{1}{2}\left(-4 \pm \sqrt{12}\right)$$

$$= \frac{1}{2}\left(-4 \pm 2\sqrt{3}\right)$$

$$= -2 \pm \sqrt{3}$$

$-2 - \sqrt{3}$ OUTSIDE OF $|z| < 1$

$-2 + \sqrt{3} \approx -.3$ IS IN

$$= \frac{2}{i}\left(2\pi i \operatorname{Res}\left(f, -2+\sqrt{3}\right)\right)$$

$$= 4\pi \left[\left. \frac{1}{\left(z - (-2 - \sqrt{3})\right)} \right|_{z = -2+\sqrt{3}} \right]$$

$$= 4\pi \left(\frac{1}{2\sqrt{3}}\right)$$

$$= \frac{2\pi}{\sqrt{3}}$$