Midterm 1

NAME:_______________________________________

RULES:
• You will be given the entire period (1PM-3:10PM) to complete the test.
• You can use one 3x5 notecard for formulas. There are no calculators nor those fancy cellular phones nor groupwork allowed. Each problem is worth 10 points, and partial credit is awarded outside of the true/false questions.
• Show all of your work. Correct answers without sufficient work will be worth nearly nothing. Also the more work you show, the easier it is for me to find your mistakes and possibly give you more points.
• Be clear what it is that you want graded - if there a multiple solutions I will grade the first one.

QUESTION 1	/ 10
QUESTION 2	/ 10
QUESTION 3	/ 10
QUESTION 4	/ 10
QUESTION 5	/ 10
QUESTION 6	/ 10
QUESTION 7	/ 10
QUESTION 8	/ 10
TOTAL	/ 80

#1. TRUE or FALSE. Each is worth one point.

T	F	STATEMENT
○	⊗	**A**. If A is a 3x1 matrix, then there could exist some matrix B such that $AB = BA$.
⊗	○	**B**. If 0 is an eigenvalue of A, then A is not invertible.
⊗	○	**C**. If A is $n \times n$, then $A + A^T$ is symmetric.
○	⊗	**D**. Every matrix with all eigenvalues nonzero can be diagonalized.
○	⊗	**E**. There is some 2x3 matrix A with an inverse.
○	⊗	**F**. If 3x3 matrices A and B have rank 2, then their sum has rank 2.
○	⊗	**G**. $\det(A + B) = \det(A) + \det(B)$.
⊗	○	**H**. If AB has an inverse, where A and B are square matrices, then A has an inverse.
⊗	○	**I**. Four vectors in $\mathbb{R}^3$ cannot be linearly independent.
○	⊗	**J**. Suppose A is 2x2 of rank 1. Then for any vector b the system $Ax = b$ has a solution.

#2. Find the eigenvalues and eigenvectors of the following matrix. Can it be diagonalized? If so, write out the matrices P and D. $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

First we find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 3 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2$$

So then $\lambda = 1, 2, 2$. We can diagonalize as long as we have 2 linearly independent eigenvectors corresponding to $\lambda = 2$. So we find how many we have by solving $(A - 2I)v = 0$:

$$\left(\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So we have 2 parameters and thus 2 linearly independent eigenvectors given by the one equation $-x + 3y + z = 0$. In other words, we can diagonalize! So $x = 3y + z$ and we can choose any vector of the form $\begin{pmatrix} 3y+z \\ y \\ z \end{pmatrix}$. In particular we'll choose

$$v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Then we find one eigenvector corresponding to $\lambda = 1$ given by solving the system $(A - I)v = 0$: $\begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This tells us that $y = z = 0$ so then our only vectors are of the form $v = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ so we will choose

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

From this our matrices P and D will be:

$$P = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

#3. Suppose that we know the following determinant: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$. Then what is the value of $\begin{vmatrix} 5g & 5d+15a & 5a \\ h & e+3b & b \\ i & f+3c & c \end{vmatrix}$?

$$\begin{aligned}
\begin{vmatrix} 5g & 5d+15a & 5a \\ h & e+3b & b \\ i & f+3c & c \end{vmatrix} &= 5 \cdot \begin{vmatrix} g & d+3a & a \\ h & e+3b & b \\ i & f+3c & c \end{vmatrix} \\
&= 5 \cdot \begin{vmatrix} g & d & a \\ h & e & b \\ i & f & c \end{vmatrix} \\
&= 5 \cdot \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \\
&= -5 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \\
&= -5 \cdot 2 = -10
\end{aligned}$$

#4. Are the vectors $(1, 3, -1), (2, 1, 0)$, and $(8, -1, 2)$ linearly independent or not? (Show work)
We put the vectors as the rows of a matrix and compute the rank of it. The rank will be the number of linearly independent rows, which is the number of linearly independent vectors out of the ones we started with:

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 8 & -1 & 2 \end{pmatrix} \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 - 8R_1 \end{matrix} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 0 & -25 & 10 \end{pmatrix} R_3 \mapsto R_3 - 5R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{pmatrix} R_2 \mapsto -\frac{1}{5} R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

This has rank 2 and thus the vectors are not linearly indepedent.

#5. Find the general solution to the equation:

$$y^{(4)} - 10y''' + 28y'' = 0$$

We assume the solution looks something like $y = e^{mx}$ and plug this in to get our condition on m:

$$m^4 - 10m^3 + 28m^2 = m^2(m^2 - 10m + 28) = 0$$

We use the quadratic formula to split up the second factor:

$$m = \frac{1}{2}\left[10 \pm \sqrt{100 - 4(28)}\right] = 5 \pm \sqrt{3}i$$

So then our general solution is (notice the repeated root of $m = 0$:

$$y = c_1 + c_2 x + c_3 e^{5x}\cos(\sqrt{3}x) + c_4 e^{5x}\sin(\sqrt{3}x)$$

#6. Solve the following initial value problem:

$$y'' + y' - 12y = 14e^{3x}, y(0) = 0, y'(0) = 0$$

We first find the homogeneous solution with the substitution $y = e^{mx}$:

$$m^2 + m - 12 = (m+4)(m-3) = 0$$

Thus we find that:

$$y_H = c_1 e^{-4x} + c_2 e^{3x}$$

Now to find y_P our naive guess would be $y_P = Ae^{3x}$ but this is in y_H. Thus we instead guess:

$$y_P = Axe^{3x}$$

$$y_P' = A(e^{3x} + 3xe^{3x}) = A(1+3x)e^{3x}$$

$$y_P'' = A(3e^{3x} + (1+3x)3e^{3x}) = A(6+9x)e^{3x}$$

Plugging this into the equation to find our condition on A:

$$\begin{aligned} y_P'' + y_P' - 12y_P &= 14e^{3x} \\ A(6+9x)e^{3x} + A(1+3x)e^{3x} - 12Axe^{3x} &= 14e^{3x} \\ 7Ae^{3x} &= 14e^{3x} \end{aligned}$$

Thus $7A = 14$ and $A = 2$. So our general solution is:

$$y = c_1 e^{-4x} + c_2 e^{3x} + 2xe^{3x}$$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x} + 2(1+3x)e^{3x}$$

Our initial conditions imply that:

$$y(0) = 0 = c_1 + c_2 \text{ and thus } c_2 = -c_1$$

$$y'(0) = 0 = -4c_1 + 3c_2 + 2 = -7c_1 + 2 \text{ and thus } c_1 = \frac{2}{7}, c_2 = -\frac{2}{7}$$

Thus our solution is:

$$y = \frac{2}{7}e^{-4x} - \frac{2}{7}e^{3x} + 2xe^{3x}$$

#7. Find the general solution of the following differential equation:

$$2x^2y'' + 6xy' + 2y = 0$$

We recognize that this is a cauchy-euler equation. So the substitution $y = x^m$ will work. Then we substitute in for y:

$$(2m(m-1) + 6m + 2)x^m = 0$$

So we want m's such that $2m^2 + 4m + 2 = 2(m+1)^2 = 0$, so $m = -1, -1$ is a repeated root. In the case of repeated roots in cauchy-euler, we multiply by lnx to find a second linearly independent solution:

$$y = c_1x^{-1} + c_2(lnx)x^{-1}$$

#8. Suppose a block with weight $12N$ stretches a spring $4m$. Then suppose this block is removed and replaced with another of mass $2kg$, and the spring affords a damping force numerically equivalent to 7 times the instantaneous velocity. Also suppose that an external force of $f(t) = 50e^{2t}$ is applied to the system. Find the equation of the motion of this mass on the spring if it starts from rest at the equilibrium position. The following might be useful:

$$mx'' + \beta x' + kx = f(t)$$

First we need to find the three constants in the above equation. We use Hooke's law to find k:

$$k = \frac{F}{x} = \frac{12N}{4m} = 3N/m$$

The values $\beta = 7$ and $m = 2kg$ are given (be careful to use the second block since that is the one that we're describing the motion of!). Thus our equation becomes:

$$2x'' + 7x' + 3x = 50e^{2t}$$

First we find the homogeneous solution via the subsitution $x = e^{mt}$. Then we want m's such that $2m^2 + 7m + 3 = 0$. By the quadratic formula:

$$m = \frac{1}{4}\left[-7 \pm \sqrt{49 - 24}\right] = -\frac{1}{2} \text{ or } -3$$

Thus our homogeneous solution is:

$$x_H = c_1 e^{-\frac{1}{2}t} + c_2 e^{-3t}$$

Then we now find the particular solution x_P, and guess $x_P = Ae^{2t}$ since this is not in x_H. Then we have $x_P' = 2Ae^{2t}$ and $x_P'' = 4Ae^{2t}$ and subsituting in to find A:

$$2x_P'' + 7x_P' + 3x_P = (8A + 14A + 3A)e^{2t} = 25Ae^{3t} = 50e^{2t}$$

Thus we see that $A = 2$. So our general solution (and its derivative) is:

$$x = 2e^{2t} + c_1 e^{-\frac{1}{2}t} + c_2 e^{-3t}$$

$$x' = 4e^{2t} - \frac{1}{2}c_1 e^{-\frac{1}{2}t} - 3c_2 e^{-3t}$$

To find c_1 and c_2, we impose our initial conditions. Since we're starting at rest and at the equilibrium position, that tells us that $x(0) = 0$ and $x'(0) = 0$. Then solving for c_1 and c_2:

$$x(0) = 0 = 2 + c_1 + c_2, \text{ so } c_2 = -2 - c_1$$

$$x'(0) = 0 = 4 - \frac{1}{2}c_1 - 3c_2$$

Solving these two equations should get $c_1 = -4$ and $c_2 = 2$. Thus our equation of motion is:

$$x = 2e^{2t} - 4e^{-\frac{1}{2}t} + 2e^{-3t}$$

#E. (EXTRA CREDIT) This problem is worth 10 points. There are 2 parts each of which is 5 points, of which you could receive either 0 or 5 (i.e. no partial credit).
When solving a homogeneous second order differential equation of the form

$$ay'' + by' + cy = 0$$

We do the substitution $y = e^{mx}$ and solve for m's that will work. In the case that we get two complex values for m we have $m = \alpha \pm i\beta$, where $\beta \neq 0$ and our general solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

1. Recall what we did in lecture to find our two real solutions from these complex solutions (write our proof).
2. Show that these two real solutions are linearly independent.

1. We did this in the notes in lecture 5 page 68.
2. We have $y_1 = e^{\alpha x}\cos\beta x$ and $y_2 = e^{\alpha x}\sin\beta x$. To show they are linearly independent we use the wronskian:

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ e^{\alpha x}(\alpha\cos\beta x - \beta\sin\beta x) & e^{\alpha x}(\alpha\sin\beta x + \beta\cos\beta x) \end{vmatrix} \\
&= e^{2\alpha x}\left[\beta\cos^2\beta x + \cancel{\alpha\cos\beta x\sin\beta x}\right] - e^{2\alpha x}\left[\cancel{\alpha\cos\beta x\sin\beta x} - \beta\sin^2\beta x\right] \\
&= e^{2\alpha x}(\beta)\left[\cos^2\beta x + \sin^2\beta x\right] \\
&= \beta e^{2\alpha x} \neq 0 \text{ since } \beta \neq 0
\end{aligned}$$

Thus the two solutions are linearly independent on all of $\mathbb{R}$.