Midterm 2

NAME:____

RULES:

- You will be given the entire period (1PM-3:10PM) to complete the test.
- You can use one 3x5 notecard for formulas. There are no calculators nor those fancy cellular phones nor groupwork allowed. Each problem is worth 10 points, and partial credit is awarded outside of the true/false questions.
- Show all of your work. Correct answers without sufficient work will be worth nearly nothing. Also the more work you show, the easier it is for me to find your mistakes and possibly give you more points.
- Be clear what it is that you want graded - if there a multiple solutions I will grade the first one.

QUESTION 1	/ 10
QUESTION 2	/ 10
QUESTION 3	/ 10
QUESTION 4	/ 10
QUESTION 5	/ 10
QUESTION 6	/ 10
QUESTION 7	/ 10
QUESTION 8	/ 10
TOTAL	/ 80

#1. Find two values of r such that there exist solutions of the form $\sum c_n x^{n+r}$ centered at the singular point $x = 0$ of the following differential equation:

$$2xy'' - y' + 2y = 0$$

For each r find the recurrence relation for the coefficients of the corresponding solution.

Since we need to find the recurrence relation we use the long way to find the r's. We make the following substitutions:

$$y = \sum_{n\geq 0} c_n x^{n+r} \qquad y' = \sum_{n\geq 0}(n+r)c_n x^{n+r-1} \qquad y'' = \sum_{n\geq 0}(n+r-1)(n+r)c_n x^{n+r-2}$$

Plugging this into the equation we get:

$$\sum_{n\geq 0} 2(n+r-1)(n+r)c_n x^{n+r-1} - \sum_{n\geq 0}(n+r)c_n x^{n+r-1} + \sum_{n\geq 0} 2c_n x^{n+r} = 0$$

$$\left[\underbrace{\sum_{n\geq 0} 2(n+r-1)(n+r)c_n x^{n-1}}_{\text{let k = n - 1}} - \underbrace{\sum_{n\geq 0}(n+r)c_n x^{n-1}}_{\text{let k = n - 1}} + \underbrace{\sum_{n\geq 0} 2c_n x^n}_{\text{let k = n}}\right] x^r = 0$$

Now we omit the x^r and continue:

$$\sum_{k\geq -1} 2(k+r)(k+r+1)c_{k+1}x^k - \sum_{k\geq -1}(k+r+1)c_{k+1}x^k + \sum_{k\geq 0} 2c_k x^k = 0$$

Now we pull out the first term out of the first two sums and combine:

$$[2(r-1)r - r]c_0 x^{-1} + \sum_{k\geq 0}[(k+r+1)(2k+2r-1)c_{k+1} + 2c_k]x^k = 0$$

We find the r's by setting the coefficient of the first term equal to zero:

$$2(r-1)r - r = 2r^2 - 3r = r(2r-3) = 0$$

So $r_1 = 0$ and $r_2 = \frac{3}{2}$. We obtain the recurrence relations by setting the coefficient of the k-th power term equal to zero:

$$c_{k+1} = \frac{-2c_k}{(k+r+1)(2k+2r-1)}$$

So our two recurrence relations are:

$$r_1 : c_{k+1} = \frac{-2c_k}{(k+1)(2k-1)} \quad \Big| \quad r_2 : c_{k+1} = \frac{-2c_k}{(k+\frac{5}{2})(2k+2)} = \frac{-2c_k}{(2k+5)(k+1)}$$

#2 Find the singular points of the following differential equation:

$$2(x-1)^2x^2(x+3)^3y'' + xy' + 2x^2y = 0$$

Classify each as regular or irregular. For each regular singular point x_0, find the numbers r such that we can find series solutions of the form $\sum c_n(x-x_0)^{n+r}$.
We put the equation in standard form:

$$y'' + \frac{y'}{2(x-1)^2(x+3)^3x} + \frac{y}{(x-1)^2(x+3)^3} = 0$$

Thus we have $x = 1, -3$ being irregular singular points and $x = 0$ being regular. We find the r's using the formula:

$$r(r-1) + a_0r + b_0 = 0$$

So we have:

$$\tilde{P} = xP = \frac{1}{2(x-1)^2(x+3)^3} \qquad a_0 = \tilde{P}(0) = \frac{1}{54}$$

$$\tilde{Q} = x^2Q = \frac{x^2}{(x-1)^2(x+3)^3} \qquad b_0 = \tilde{Q}(0) = 0$$

So we plug these in:

$$r(r-1) + \frac{1}{54}r = r(r - \frac{53}{54}) = 0$$

Thus $r = 0$ or $r = \frac{53}{54}$.

#3 Find the general solution to the following system of differential equations:

$$X' = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} X$$

First we find the eigenvalues (expanding along the bottom row):

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -2 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (2-\lambda)[(1-\lambda)^2+4] = 0$$

This should get us $\lambda_1 = 2$, $\lambda_2 = 1+2i$, $\lambda_3 = 1-2i$. We now find eigenvectors:

$$(A-2I)v_1 = 0$$

$$\left(\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_2 \mapsto R_2 - 2R_1 \left(\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So looking at the second equation $5y = 7z$ so choose $z = 5$ so that $y = 7$. Then the first equation yields $x = 4$. Thus we have:

$$v_1 = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$

Now we find one of the complex eigenvectors:

$$(A-(1+2i)I)v_2 = 0$$

$$\left(\begin{array}{ccc|c} -2i & 2 & -2 & 0 \\ -2 & -2i & 3 & 0 \\ 0 & 0 & 1-2i & 0 \end{array}\right)$$

The last equation implies $z = 0$ and we should find that:

$$\lambda_2 = \underbrace{1}_{\alpha} + \underbrace{2}_{\beta} i \qquad v_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{B_1} + i \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{B_2}$$

Thus we can plug these in to our general solution with the following substitutions:

$$\alpha = 1 \qquad \beta = 2 \qquad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

This will get our real general solution to be:

$$X = c_1 e^{2t}\begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} + c_2 e^t \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right] + c_3 e^t \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t\right]$$

#4 Consider the system of differential equations:

$$X' = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix} X$$

Find an initial condition $X(0)$ so that the solution goes to the origin as $t \to \infty$ and another initial condition so that the solution diverges as $t \to \infty$. By diverge I mean one of the x or y components of our solution approaches ∞.
Hint: Start with the general solution.
First we find the eigenvalues:

$$\begin{vmatrix} -4-\lambda & -3 \\ 2 & 3-\lambda \end{vmatrix} = (-4-\lambda)(3-\lambda) + 6 = (\lambda+3)(\lambda-2) = 0$$

Thus we have that $\lambda_1 = 2$ and $\lambda_2 = -3$. We find the eigenvectors:

$$(A - 2I)v_1 = 0$$

$$\left(\begin{array}{cc|c} -6 & -3 & 0 \\ 2 & 1 & 0 \end{array}\right) R_2 \mapsto R_2 + \frac{1}{3}R_1 \left(\begin{array}{cc|c} -6 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So our equation is $-6x - 3y = 0$ so let $x = 1$ and then $y = -2$, so $v_1 = \binom{1}{-2}$.

$$(A + 3I)v_2 = 0$$

$$\left(\begin{array}{cc|c} -1 & -3 & 0 \\ 2 & 6 & 0 \end{array}\right) R_2 \mapsto R_2 + 2R_1 \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So we can let $v_2 = \binom{-3}{1}$. Thus our general solution is:

$$X = c_1 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Now recall that the initial condition $X(0)$ determine the constants c_1 and c_2 and vice-versa. So let's choose c_1 and c_2 so that we have a solution that goes to the origin ($c_1 = 0, c_2 = 1$). Then our solution is:

$$X = e^{-3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

So in this case we start at $X(0) = \binom{-3}{1}$, which is the first answer. Now to find a solution with a divergent x or y coordinate, we need only have $c_1 \neq 0$ since then we will have exponential growth in both x and y. So choose $c_1 = 1, c_2 = 0$ and get:

$$X = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

In this case our we start at $X(0) = \binom{1}{-2}$. These two initial conditions will work for this question (there are MANY you can choose). If you recall how we pictured our solutions in a problem very similar to this in class (see lecture

8 notes page 104) we had our solution paths along the lines containing the eigenvectors being stuck on those lines. If the eigenvalue was < 0 the paths converged to the origin, and if the eigenvalue was > 0 the paths went out to infinity. So if you remembered this picture you would immediately know that if we start at these eigenvectors we get the desired properties.

#5 Compute the following line integral:

$$\int_C \frac{\sin 2y - 2y}{4} + x^2 dx + x\cos^2 y dy$$

Where C is the curve around the circle radius one centered at the origin from (1,0) to (-1,0) traversed counterclockwise.
Here we initially notice that direct computation will be too difficult. Then we check for a potential function but find that the vector field

$$F = (P, Q) = (\frac{\sin 2y - 2y}{4} + x^2, x\cos^2 y)$$

is not equal to the gradient of some function. This can be seen by checking that $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. So our only option is to try to use Green's theorem. For this we complete our semicircle arc to a loop around a half-circle with the path $r(t) = (t, 0)$ from $t =$-1 to 1. Let's call this path $\tilde{C}$, the resulting closed area R, and F will be as above. Then Green's theorem reads:

$$\int_C F \cdot dr + \int_{\tilde{C}} F \cdot dr = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$$

So we compute the other two terms in the above equation. First on $\tilde{C}$ we use $r(t) = (t, 0)$ so $r'(t) = (1, 0)$:

$$\int_{\tilde{C}} F \cdot dr = \int_{\tilde{C}} F \cdot r'(t)dt = \int_{-1}^{1} (t^2, t) \cdot (1, 0)dt = \int_{-1}^{1} t^2 dt = \frac{2}{3}$$

Next we compute the double integral:

$$\begin{aligned}
\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy &= \int\int_R \cos^2 y - \frac{1}{4}(2\cos 2y - 2)dxdy \\
&= \int\int_R \frac{1}{2}(1 + \cos 2y) - \frac{1}{2}(\cos 2y - 1)dxdy \\
&= \int\int_R dxdy \\
&= \text{Half the area of the unit circle} \\
&= \frac{\pi}{2}
\end{aligned}$$

Thus our answer is the difference of these two:

$$\int_C F \cdot dr = \frac{\pi}{2} - \frac{2}{3}$$

#6 Compute the work integral $\int_C F \cdot dr$ if the path C is given by

$$r(t) = (\cos t, t, t^2) \text{ from } t = 0 \text{ to } t = \pi$$

with the vector field $F = (z + e^y, xe^y, x)$.
Here we find that $F = \nabla\phi$ for some ϕ:

$$\phi = \int z + e^y dx = xz + xe^y + C_1(y, z)$$

$$\phi = \int xe^y dy = xe^y + C_2(x, z)$$

$$\phi = \int xdz = xz + C_3(x, y)$$

We see that $\phi = xz + xe^y$ satisfies $\nabla\phi = F$. Thus our integral only depends on the endpoints $r(\pi) = (-1, \pi, \pi^2)$ and $r(0) = (1, 0, 0)$:

$$\int_C F \cdot dr = \phi(r(\pi)) - \phi(r(0)) = -\pi^2 - e^\pi - 1$$

#7 Compute the flux integral $\int\int F \cdot ndS$ along the part of the paraboloid

$$z = 9 - x^2 - y^2 \text{ with } z \geq 0$$

Let $F = (x - y, x + y, xy)$. Use the unit normal with positive z-coordinate. We begin with parametrizing the surface and finding the normal vector:

$$\sigma = (x, y, 9 - x^2 - y^2)$$

$$\sigma_x = (1, 0, -2x)$$
$$\sigma_y = (0, 1, -2y)$$
$$\sigma_x \times \sigma_y = (2x, 2y, 1)$$

This normal vector has a positive z-coordinate and thus will give us the correct unit normal $n = \frac{\sigma_x \times \sigma_y}{||\sigma_x \times \sigma_y||}$. Then we substitute this and $dS = ||\sigma_x \times \sigma_y||dxdy$ into our integral:

$$\begin{aligned}
\int\int F \cdot ndS &= \int\int F \cdot \frac{\sigma_x \times \sigma_y}{\cancel{||\sigma_x \times \sigma_y||}} \cancel{||\sigma_x \times \sigma_y||} dxdy \\
&= \int\int (x - y, x + y, xy) \cdot (2x, 2y, 1) dxdy \\
&= \int\int 2x^2 - \cancel{2xy} + \cancel{2xy} + 2y^2 + xydxdy \\
&= \int\int 2(x^2 + y^2) + xydxdy \qquad \text{convert to polar} \\
&= \int_0^{2\pi} \int_0^3 r^3[2 + \cos\theta \sin\theta] dr d\theta \\
&= \int_0^{2\pi} \frac{1}{4} r^4|_0^3 [2 + \cos\theta \sin\theta] d\theta \\
&= \frac{81}{4}[2\theta + \frac{1}{2}\sin^2\theta]_0^{2\pi} \\
&= \frac{81}{4}(4\pi) \\
&= 81\pi
\end{aligned}$$

We knew to integrate over the circle radius 3 centered at the origin since that is where our downward-opening paraboloid intersects the xy-plane (just set $z = 0$).

#8 Compute the circulation integral $\oint_C F \cdot dr$ along the curve C given by $r(t) = (4\cos t, 2\sin t, 16)$ with vector field $F = (3y, 2x, 2y^2 + \frac{1}{2}x^2)$ using Stokes' theorem as follows. Consider C as the boundary of the paraboloid $z = x^2 + 4y^2$ with $z \leq 16$. Now compute the circulation using a surface integral.
Hint: The area of the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.
We start by computing curl$F$:

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 2x & 2y^2 + \frac{1}{2}x^2 \end{vmatrix} = (4y, -x, -1)$$

Now we parametrize the surface:

$$\sigma = (x, y, x^2 + 4y^2)$$

$$\sigma_x = (1, 0, 2x)$$

$$\sigma_y = (0, 1, 8y)$$

$$\sigma_x \times \sigma_y = (-2x, -8y, 1)$$

Since we're traversing the boundary curve in a counterclockwise direction looking down from the positive z-axis, using the righthand rule know we want to use the unit normal with positive z-coordinate. So we have the correct sign and $n = \frac{\sigma_x \times \sigma_y}{||\sigma_x \times \sigma_y||}$. Thus our integral becomes:

$$\begin{aligned}
\oint_C F \cdot dr &= \int\int \text{curl}F \cdot n dS \\
&= \int\int \text{curl}F \cdot \frac{\sigma_x \times \sigma_y}{\cancel{||\sigma_x \times \sigma_y||}} \cancel{||\sigma_x \times \sigma_y||} dxdy \\
&= \int\int (4y, -x, -1) \cdot (-2x, -8y, 1) dxdy \\
&= -\int\int dxdy \\
&= -\text{Area of the ellipse with equation} \frac{x^2}{4^2} + \frac{y^2}{2^2} = 1 \\
&= -\pi(2)(4) \\
&= -8\pi
\end{aligned}$$

We found our integration domain to be that ellipse since it is when $z = 16$ in our upward-opening paraboloid. So we get this cross section is given by the equation $16 = x^2 + 4y^2$ which is precisely that ellipse.