CHAPTER 13.2 PROBLEM 37

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We will *separate the variables r and t* in the following problem. Separation of variables extends to more general sorts of differential equations, but we spell out a simple application of the technique here.

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First define x, y so that $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. The differential equation then becomes:

$$x'' = -kx',\ y'' = -g - ky'.$$

Integrating both sides of both equations, we get

$$x' = -kx + C_1,\ y' = -gt - ky + C_2$$

for some constants C_1, C_2. Now letting

$$v_x = -kx + C_1, \quad v_y = -gt - ky + C_2,$$

we see $dx/dt = v_x,\ dy/dt = v_y$ and hence rearranging we get

$$dx/v_x = dt,\ dy/v_y = dt.$$

Integrating both sides of both equations, we get

$$\int dx/v_x = \int dt,\ \int dy/v_y = \int dt.$$

The x-coordinate. Expanding the left and right sides of the first equation,

$$\int dx/v_x = -k^{-1} \int {}^{dx}\!/_{x} = -k^{-1} \ln(x - C_1 k^{-1}) + D_1, \quad \int dt = t + C$$

for constants D_1, C. Solving for x we get

$$x(t) = E_1 e^{-kt} + F_1$$

for constants E_1, F_1 ($E_1 = e^{C-D_1}$, $F_1 = C_1 k^{-1}$). From our initial conditions,

$$E_1 + F_1 = x(0) = 0,\ -kE_1 = -kE_1 e^{-k0} = x'(0) = v_0 \cos\alpha$$

hence $E_1 = -k^{-1} v_0 \cos\alpha$ and $F_1 = k^{-1} v_0 \cos\alpha$, and hence

$$x(t) = -k^{-1} v_0 \cos\alpha e^{-kt} + k^{-1} v_0 \cos\alpha.$$

The y-coordinate. Similarly as before, we can deduce that

$$y(t) = E_2 e^{-kt} + F_2 t^{-1} + G_2$$

for constants E_2, F_2, G_2, use the initial conditions to conclude

$$E_2 = -k^{-1} v_0 \sin\alpha - k^{-2} g,\ F_2 = -gk^{-1},\ G_2 = k^{-1} v_0 \sin\alpha + k^{-2} g,$$

plug in those values to get an explicit form for $y(t)$ in terms of t and simplifying. You should spell this step out on your own for practice with the technique.