LAGRANGE MULTIPLIERS: MULTIPLE CONSTRAINTS

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Our motivation is to deduce the diameter of the semimajor axis of an ellipse non-aligned with the coordinate axes using Lagrange Multipliers. Therefore consider the ellipse given as the intersection of the following ellipsoid and plane:

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{25} = 1$$
$$x + y + z = 0$$

Note that the center of the ellipsoid is $(0,0,0)$ and hence the center of the elliptical intersection of the ellipsoid and plane is also $(0,0,0)$. The semimajor axis has half of its diameter given by the maximum distance between $(0,0,0)$ and a point on the ellipse. Therefore we wish to maximize the function

$$f(x,y,z) = x^2 + y^2 + z^2$$

on the above ellipse. Let

$$g(x,y,z) = \frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{25}, \quad h(x,y,z) = x + y + z.$$

We wish to solve for $x, y, z, \lambda_1, \lambda_2$ satisfying the system

$$g(x,y,z) = 1$$
$$h(x,y,z) = 0$$
$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$$

Noting that $\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle x, {}^{y}/_{2}, {}^{2z}/_{25} \rangle$, $\nabla h = \langle 1,1,1 \rangle$,

$$2x = \lambda_1 x + \lambda_2$$
$$2y = \lambda_1 y + \lambda_2$$
$$2z = 2\lambda_1 {}^{z}/_{25} + \lambda_2$$

Rearranging, we obtain

$$x(2 - \lambda_1) = y(2 - \lambda_1) = z(2 - {}^{2\lambda_1}/_{25}) = \lambda_2$$

In the case $\lambda_1 \neq 2$, $y = x$, $z = -2x$ by $h = 0$, $x = \pm{}^{\sqrt{3}}/_{2}$ by $g = 1$, hence

$$(x,y,z) = ({}^{\sqrt{3}}/_{2}, {}^{\sqrt{3}}/_{2}, -\sqrt{3}), (-{}^{\sqrt{3}}/_{2}, -{}^{\sqrt{3}}/_{2}, \sqrt{3})$$

In the case $\lambda_1 = 2$, $\lambda_2 = 0$, hence $2z = 4{}^{z}/_{25}$, hence $z = 0$, hence $x = -y$ by $h = 0$, hence $x = \pm 1$ hence

$$(x,y,z) = (1,1,0), (-1,-1,0).$$

Checking all four points, we observe that f attains a maximum value of ${}^{9}/_{2}$. Hence the full diameter of the semimajor axis is 9.