SOME NOTES ON STOKES THEOREM

MATH 114-003

1. Flux Integrals

Recall from last class that we introduced integrals of the following form:

$$\int\int_S \mathbf{G} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{G}$ is a vector-valued function $S \to \mathbb{R}^3$, $\mathbf{n}$ denotes the normal vector to the surface S pointing in the direction of the orientation of S (default: outward), and $d\sigma$ denotes the area element for surface integrals (analogous to the ds used in line integrals).

When the surface S is smoothly parametrized by a function $\mathbf{r} : [a, b] \times [c, d] \to S$, then $\mathbf{G}\cdot\mathbf{n}\, d\sigma$ simplifies to either $\mathbf{G}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$ or $\mathbf{G}\cdot(\mathbf{r}_v \times \mathbf{r}_u)\, du\, dv$, depending on whether $\mathbf{r}_u \times \mathbf{r}_v$ or $\mathbf{r}_v \times \mathbf{r}_u$ is pointing in the same direction as the outward normal $\mathbf{n}$.

2. Stokes Theorem

We defined the *curl* of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ to be the vector field

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

We briefly introduced Stokes' Theorem at the end of the last lecture:

Theorem 2.1. *We have*

$$\int_C \mathbf{F}\cdot\ dr = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma,$$

for a piecewise smooth oriented surface S having as its boundary a piecewise smooth curve C, a vector field $\mathbf{F}$ having continuous partial derivatives defined on an open region containing S.

This generalizes Green's Theorem for surfaces in $\mathbb{R}^3$. As before, we follow the convention that the boundary C of a surface S is oriented in such a way so that if you are travelling along the orientation of C, S will be on your left and curling your fingers in the direction of C will point the thumb in the orientation of S.

3. Verification of Stokes' Theorem

Let's verify Stokes' Theorem for

$$\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$$

over the interior S of the ellipse C defined as the intersection of the plane $z = x$ with the cylinder $x^2 + y^2 = 1$. (Imagine a right circular cyclinder sliced by a plane tilted at $\pi/4$ radians.)

3.1. **Line integral.**

A parametrization of the ellipse C is

$$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, \cos\theta \rangle, \quad \theta \in [0, 2\pi)$$

Hence we calculate

$$\frac{d}{dt}(\mathbf{r})(\theta) = \langle -\sin\theta, \cos\theta, -\sin\theta \rangle.$$

Hence the line integral

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \frac{d}{dt}(\mathbf{r})(\theta)\, d\theta \\ &= \int -\sin(\theta)\cos(\theta) + 1 - 3\sin^2(\theta)\, d\theta \\ &= \left[\frac{\sin^2(\theta)}{2} + \theta - \frac{3\theta}{2} + \frac{3\sin 2\theta}{4}\right]_0^{2\pi} \\ &= -\pi.\end{aligned}$$

3.2. **Surface Integral.**

A parametrization of the surface S is

$$\mathbf{r}(u, v) = \langle u, v, u \rangle, \quad u^2 + v^2 \leqslant 1.$$

Hence we calculate

$$(\nabla \times \mathbf{F}) = \ldots = \mathbf{i}, \quad \mathbf{r}_u \times \mathbf{r}_v = \mathbf{k} - \mathbf{i}.$$

Hence the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int\int_{u^2+v^2\leqslant 1} -1\, du\, dv = -\pi.$$

4. Verification on an Implicitly Defined Surface

Let's verify Stokes' Theorem for

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$$

over the surface $S = \{(x, y, z) \mid x^2 + y^2/4 + z^2/a^2, \quad z \geqslant 0\}$ oriented upwards. First of all, what does S look like? What does the boundary C look like? Notice that S depends on a but C does not depend on a.

4.1. **Surface Integral.** Let's say we didn't want to parametrize the surface S. The surface S is a level set of the function G defined by $G(x,y,z) = x^2+y^2/4+z^2/a^2 = 1$. Therefore the outward normal $\mathbf{n}$ to the surface is one of the two choices

$$\frac{\nabla G}{|\nabla G|} = \frac{\langle 2x, y/2, 2z/a^2\rangle}{|\nabla G|}, \quad -\frac{\nabla G}{|\nabla G|} = -\frac{\langle 2x, y/2, 2z/a^2\rangle}{|\nabla G|}.$$

Since S is oriented upwards, $\mathbf{n}$ is the first choice. The curl

$$\nabla \times \mathbf{F} = 2\mathbf{k}.$$

Therefore

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = 4z/a^2.$$

Also note that $G_z \neq 0$ for $z > 0$, hence we can regard S as the graph of a function of x, y, hence

$$d\sigma = \frac{|\nabla G|}{|G \cdot \mathbf{k}|} dx\, dy$$

and hence the integrant simplifies to

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = ... = 2.$$

Therefore the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int\int_{\{x^2+2y^2 \leqslant 1\}} 2 = 4\pi.$$

4.2. **Line integral.** The boundary C of S is the ellipse $x^2 + y^2/4 = 1$ oriented counterclockwise, hence a parametrization of C is given by the path

$$\mathbf{r}(\theta) = \langle \cos\theta, 2\sin\theta\rangle, \quad \theta \in [0, 2\pi)$$

and the line integral becomes

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_0^{2\pi} ... = 4\pi$$

5. The theorem as a Computational Shortcut

5.1. **Simplifying Surfaces.** One consequence of Stokes' Theorem is that *the surface integral of the form* $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}$ *only depends on the boundary of* S. So if we don't like S, we can change it. In the previous example, the surface was $S = \{(x,y,z) \mid x^2 + y^2/4 + z^2/a^2 = 1, \quad z \geqslant 0\}$. We could have replaced S with the surface $T = \{(x,y,0) \mid x^2 + y^2 \leqslant 1\}$ and gotten the same answer....

5.2. **Changing a Line Integral to a Surface Integral.** Let C be the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ oriented counter-clockwise. Then S is a filled in triangle whose orientation contains all positive components. To calculate

$$\int_C \mathbf{F} \cdot \mathbf{dr},$$

you need to break up C into three lines and compute three integrals. To calculate

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{c}\, d\sigma,$$

you just need to compute a single double integral after a suitable parametrization. For a concrete example: let $\mathbf{F} = \langle z^2, y^2, x\rangle$. The integrals compute to $-1/6$...