TRIPLE INTEGRALS FROM LECTURE

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Integrating over a cylinder sliced by a plane

Let D be the region bounded by the surfaces

$$z = 0,\ x^2 + 4y^2 = 4,\ z = x + 2.$$

Then the volume of D is given by a triple integral

$$\int\int\int_D dV,$$

whose bounds of integration we calculate below as follows. The projection of D onto the yz-plane is given by the region $(y, z) \in [-1, +1] \times [0, 4]$. Hence our innermost integral will have bounds from some function $h_1(y, z)$ to some function $h_2(y, z)$. In particular, the minimum x-value occurs on the plane $z = x + 2$ and the maximum x-value occurs on the cylinder $x^2 + 4y^2 = 4$. Hence $h_1(y, z) = z - 2$ and $h_2(y, z) = \sqrt{4 - 4y^2} = 2\sqrt{1 - y^2}$. Hence the triple integral becomes

$$\begin{aligned}
\int\int\int_D dV &= \int_0^4 \int_{-1}^1 \int_{z-2}^{2\sqrt{1-y^2}} 1\ dx\ dy\ dz \\
&= \int_0^4 \int_{-1}^1 (2\sqrt{1-y^2} + (2 - z))\ dy\ dz \\
&= \int_0^4 \left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^{+1} + 2(2 - z)\ dz \\
&= 4\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^{+1} + 4 - \left[z^2\right]_0^4\ dz \\
&= 4\pi.
\end{aligned}$$

To get the third equality, we use that an antiderivative for $\sqrt{1 - y^2}$ is

$$(\arcsin y)/2 + y\sqrt{1-y^2}/2,$$

which you can derive as in class by either of the trigonometric substitutions $y = \cos\ u$ or $y = \sin u$.

A weirder region

Let D be the region bounded by the surfaces

$$x = 1 - y^2,\ x = 0,\ z = 0,\ z = x^2 + y^2.$$

By projecting D onto the xy-plane, we similarly calculate the volume

$$\begin{aligned}\int\int\int_D dV &= \int_0^1 \int_{-\sqrt{1-x}}^{+\sqrt{1-x}} \int_0^{x^2+y^2} dz\ dy\ dx \\ &= \int_0^1 \int_{-\sqrt{1-x}}^{+\sqrt{1-x}} x^2+y^2\ dy\ dx \\ &= \int_0^1 2x^2\sqrt{1-x} + \left[{}^{y^3}\!/_3\right]_{-\sqrt{1-x}}^{+\sqrt{1-x}}\ dx \\ &= \int_1^0 2x^2\sqrt{1-x}\ dx + \int_0^1 {}^2\!/_3(1-x)^{3/2}\ dx \\ &= \left[-{}^4\!/_{105}(1-x)^{3/2}(3x(5x+4)+18)\right]_0^1 + \left[-{}^4\!/_5(1-x)^{3/2}\right]_0^1 \\ &= {}^{156}\!/_{105}.\end{aligned}$$

To get the second to last equality, we use that an antiderivative of $x^2\sqrt{1-x}$ is

$$-{}^2\!/_{105}(1-x)^{3/2}(3x(5x+4)+18).$$

Some antiderivatives

antiderivative for $\sqrt{1-y^2}$: Set $y = \sin u$ and observe

$$\begin{aligned}\int \sqrt{1-y^2}\ dy &= -\int \sqrt{1-\sin^2 u}(\cos u)\ du \\ &= \int {}^1\!/_2 + {}^{\cos 2u}\!/_2\ du \\ &= {}^u\!/_2 + {}^{\sin 2u}\!/_4 \\ &= {}^{\arcsin y}\!/_2 + {}^{\sin 2\arcsin y}\!/_2 \\ &= {}^{\arcsin y}\!/_2 + {}^{(\sin\arcsin y)(\cos\arcsin y)}\!/_2 \\ &= {}^{\arcsin y}\!/_2 + {}^{y\sqrt{1-y^2}}\!/_2\end{aligned}$$

antiderivative for $\sqrt{1-y^2}$: The relevant substitution is

$$u^2 = 1-x$$

so that $2u\ du = -x\ dx$. The calculation then simplifies to finding the antiderivative of a polynomial.