Perimeters, Areas, and a Long Snowflake

1. Introduction

Let's say you are starting a vegetable garden. You want to put a fence around it, but you only have 40 feet of fence to use. You want to biggest garden you can make, so that you can grow as many vegetables as possible. Are there different ways to arrange the fence around the perimeter so that you get a bigger area inside the fence? This problem investigates area and perimeter. You might think that if one shape has more area than another, then it is "bigger". In what ways is this true? Do bigger shapes have longer perimeters? On the other hand, can you find shapes that have the same perimeter but different areas? Today you will discover the answer to these questions and find out a few other interesting facts about the relationship between area and perimeter.

2. Materials

- Lots of square tiles
- Tracing paper
- Pencil and paper
- Triangular grid paper
- Graph paper

3. Vocabulary

- area
- fractal
- perimeter

4. What Perimeters Have the Same Area?

Count out 16 squares, and assume that the length of the side of the square is 1. Arrange these squares into a larger square. What is the perimeter of the larger square?

Try to arrange these 16 squares into shapes that have perimeters of length:

(1) 20

(2) 34

(3) 18

Sketch your results here:

Do these shapes all have the same area? (If so, what is it?)

4.1. What Areas Have the Same Perimeter? Count out 9 triangles, and assume that the length of the side of a triangle is 1. Arrange them to form any shape you want.

(1) What is the perimeter of the shape you made?

(2) Take away 2 of the triangles. Can you arrange the remaining triangles into a shape that has the same perimeter? In other words, can you find a shape with the same perimeter but smaller area?

(3) Can you arrange *more* than 9 triangles into a shape that has the same perimeter as your original large triangle?

5. The Koch Snowflake

So far, you have found that there are shapes that have the same area and different perimeter. You also found shapes that have the same perimeter and different area. How different can the area and perimeter be? You'll investigate this question further in this part of the lab. You'll need to have the page with the triangular grid and the page with the blank table handy. On the triangular grid, the smallest trianlge has side length 1.

First Step: Draw an equilateral triangle (step 1) on the triangular grid so that each side has length 9. In the table, fill in the number of sides, the length of a side, and the total perimeter. Also fill in the area of the triangle.

Second Step: You can make step 2 out of step 1 by dividing each side of the triangle into thirds and replacing the middle third by a triangle with side length 3. Each of the new segments has the same length.

(1) How many sides are there in the new shape?

(2) What is the length of each side?

(3) What is the perimeter of the new shape?

(4) What is the area of the new shape?

Fill in the second row of the table with your answers.

Third Step: For step 3, divide each side into thirds. Replace the middle third of each side with a triangle in the same way that Step 2 was made out of Step 1. Answer the questions in Step 2 again, and fill in the third row of the table with your answers.

More Steps: Repeat your actions in the third step on the shape you made in the third step. Answer the questions in Step 2 again, and fill in the rows of the table with your answers (you'll probably want to stop drawing after the third or fourth step and just figure out what the numbers in the rows are going to be).

Make a Graph: Make a graph of the perimeter of the first eight shapes. Make another graph of the area of the first eight shapes. Describe what is happening to the perimeter and the area.

If you continue this process infinitely many times, you get what is called a fractal.

# of steps	# of sides	Length of side	**Perimeter**	**Area**
1				
2				
3				
4				
5				
6				
7				
8				

6. Constructing fractals using the computer

(1) Go to the website http://compute2.shodor.org/snowflake
(2) Change the number of iterations to 1. (This lets you see one step of the fractal construction at a time.)
(3) Click on “Grid Background” and “Snap Active Point To Grid” (this lets you space the points evenly.)
(4) Use the mouse to drag your “active point” to anywhere you want it.
(5) Choose another active point and drag it anywhere you want.
(6) Continue placing the points until you have a picture you want to make into a fractal.
(7) Click on the button “up one” to see what the first “iteration” of the fractal looks like.
(8) Continue clicking “up one” to see your fractal evolve!

6.1. Your fractal.

(1) Did your fractal start to look like a curve instead of a bunch of straight lines? If so, at what iteration?

(2) Did your fractal ever intersect itself? (Why or why not?)

7. The Mandelbrot Set and some fractals in nature

- The Mandelbrot set is a fractal that models the "self-similarity" in the molecular behavior of blood vessels.
- Take a look at the webpage http://compute2.shodor.org/cgi-bin/mandy/cnew.pl
- Click on a part of the picture to zoom in (like on mapquest) on that part.
- Notice that unlike mapquest, the picture looks pretty much the same as the zoomed out version. This is "self-similarity". It means that
- Check out the following websites to see examples of fractals in nature:

  classes.yale.edu/fractals/atma/FracNat/FracNat.html

  www.phys.uni.torun.pl/ duch/zdjecia/00Siberia/syb1.html

  www.fractal.org/Fractal-Research-and-Products/Dissecting-fractals.htm

  www.photo.net/photodb/photo?photo_id=1236856

  www.bendov.info/cours/sources/fracs/fraceng/sld008.htm

8. Teaching Notes

- Make sure that the Penn students get across (or ensure that their students understand) that the number of tiles in the first two parts of the activity correspond to area: taking away tiles reduces area; adding tiles increases area.
- Bring up the definition of area with the Penn students. How can the area of an irregular shape be defined (without calculus)? The point in the lab is that area should not be thought of as the result of a calculation, but as an "amount of space that is covered" — but what does that *mean*?
- Discuss stopping points in this lab explicitly, as it is long.

9. Penn Introduction

The purpose of this lab is to investigate the difference between perimeter and area. A knee-jerk reaction to the question of whether the area grows as perimeter does would be "yes — the shape is getting larger!" But, so long as we do not restrict our attention to one fixed shape, the answer is a resounding "no". Your students will explore why this is true in the first part of the lab, in which they will find different shapes that:

- have the same area but different perimeter; and
- have the same perimeter but different area.

Some reasoning will be required to make general statements about the relationship (or lack thereof) between perimeter and area.

They will then start examining a "squared-off" version of the Koch snowflake (a beautiful object in its own right, the Koch snowflake is an example of a fractal, i.e. an object that is made up of parts similar to the whole on ever smaller scales; we'll talk more about these in class). The original Koch snowflake is constructed iteratively by attaching smaller triangles to the sides of an equilateral triangle, and then attaching still smaller triangles to the sides of the resulting figure, and so on. Your students will perform the first few iterations of the construction for the squared-off version, and will investigate what happens to the area and perimeter. This is an opportunity for them to practice their table- and graph-making skills. If they get far enough to look at the discussion questions, they will even be asked to think (very very intuitively) about limits!

Note: Many ideas and some details for this lab taken from NCTM's Navigating Through Geometry in Grades 9–12 *and Jens Feder's* Fractals*; the inspiration was chapter 4 of Liping Ma's* Knowing and Teaching Elementary Mathematics.