CONJUGATE POINTS ON THE DIFFEOMORPHISM GROUP AND BLOWUP OF THE 3-D EULER EQUATIONS

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1. Introduction

The theory of ideal (inviscid) incompressible fluid mechanics is one of the most mathematically beautiful theories in physics. This is partly because one does not need any parameters to describe the system: as soon as one has a Riemannian manifold M, possibly with boundary, one can construct the volume-preserving diffeomorphism group $\mathcal{D}_\mu(M)$, and on this infinite-dimensional manifold define a Riemannian metric using the kinetic energy integral. Arnold [A] showed that the geodesics of this metric on $\mathcal{D}_\mu(M)$ are precisely the ideal incompressible fluid flows on M, in the Lagrangian coordinate description. Thus, once one gets past the fairly serious technical issues of functional analysis involved in constructing a topology on $\mathcal{D}_\mu(M)$ and proving that the geodesic equations are well-posed (as accomplished by Ebin and Marsden [EMa]), one has essentially reduced much of ideal fluid mechanics to a study of geometry. Of course, this does not automatically solve the outstanding problems of fluid mechanics, but it does give a different context to them.

Perhaps the most serious outstanding open problem of ideal fluid mechanics is the global existence of solutions of the 3-D Euler equations. Local existence and uniqueness is well-known, and Ebin-Marsden [EMa] provided a new proof of this result in Sobolev spaces using Arnold's geometric approach. In this language, local existence and uniqueness is equivalent to smoothness of the exponential map $\exp\colon T_{\mathrm{id}}\mathcal{D}_\mu(M) \to \mathcal{D}_\mu(M)$. Global existence is then, in the language of Riemannian geometry, geodesic completeness of the manifold $\mathcal{D}_\mu(M)$. This question has long been open when M is 3-dimensional.

Our goal in this research is to relate the global existence question to a certain question about conjugate points on $\mathcal{D}_\mu(M^3)$. We show that the famous blowup condition of Beale, Kato, and Majda [BKM] for a solution (geodesic) η, together with some additional conjectured blowup properties that have been observed to hold in all known numerical experiments, imply that there are conjugate pairs arbitrarily close to the blowup time T. This would imply that there is no weak solution which locally minimizes energy near T, a stronger nonexistence result than is currently known.

Conjugate points on $\mathcal{D}_\mu(M)$ have been of interest ever since Arnold [A] computed the sectional curvature on $\mathcal{D}_\mu(\mathbb{T}^2)$, found that it was usually negative but sometimes positive, and asked whether one could find conjugate points. Computational difficulties

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prevented much progress in this direction, until Misiołek [M1] proved that one could construct some simple examples of conjugate points in $\mathcal{D}_\mu(S^3)$ along geodesics corresponding to rigid rotations of the 3-sphere. Here, positive curvature on the underlying manifold M helps one obtain positive curvature on $\mathcal{D}_\mu(M)$, which leads to the conjugate points. More surprisingly (and far more difficult computationally), Misiołek [M2] showed that conjugate points exist on $\mathcal{D}_\mu(\mathbb{T}^2)$, using an example similar to the one about which Arnold had asked.

Since the work of Misiołek, substantial progress has been made in understanding conjugate points on $\mathcal{D}_\mu(M)$. For example, Shnirelman [Sh] proved that the diameter of $\mathcal{D}_\mu(M^3)$ is finite for any 3-D manifold M, and using this result and the generalized flows of Brenier [B1], showed that there must be "local cut points" along any sufficiently long geodesic in $\mathcal{D}_\mu(M^3)$. Local cut points are points for which the geodesic is not minimizing between these two points, and there is an arbitrarily close path which yields a strictly shorter distance. In finite dimensions, these are conjugate points; on the infinite-dimensional manifold $\mathcal{D}_\mu(M^3)$, this may not be true. No such result is possible in the 2-D case, since the diameter of $\mathcal{D}_\mu(M^2)$ is infinite.

More recently, Ebin, Misiołek, and the author [EMP] studied the nature of the differential of the exponential map. Singularities of $d\exp$ are precisely the conjugate points, so the nature of conjugate points tells us much about the structure of the exponential map. The map $d\exp$ is a mapping from one infinite-dimensional space to another, and its singularities may be of two types: failure to be injective, and failure to be surjective. (For finite-dimensional mappings, both types always coincide.) Grossman [G] called these singularities monoconjugate points and epiconjugate points, respectively. The authors of [EMP] showed that in $\mathcal{D}_\mu(M^2)$, both types of conjugate points are the same, so that the exponential map is Fredholm. On the other hand, they also showed that in $\mathcal{D}_\mu(M^3)$, it is possible to have an epiconjugate point that is not a monoconjugate point. Their explicit example is the solid flat torus $D^2 \times S^1$, where the geodesic η is rigid, unit speed rotation of the disc. Here $\eta(\pi)$ is an epiconjugate point but not a monoconjugate point; in addition, for every $\varepsilon > \pi$, there is a $t_o \in (\pi, \pi + \varepsilon)$ such that $\eta(t_o)$ is monoconjugate to $\eta(0)$. So the structure of conjugate points on $\mathcal{D}_\mu(M^3)$ is in general much more complicated than on $\mathcal{D}_\mu(M^2)$ or on a finite-dimensional Riemannian manifold.

In Section 3, we explain why it has often been easier to find conjugate points in three dimensions than in two. It turns out that one can construct local approximations of Jacobi fields concentrated near any point in three dimensions, and use these to search for genuine Jacobi fields. In Theorem 3.1, we prove that one can construct a divergence-free vector field in a neighborhood of any point x interior to M, such that the index form along a geodesic in $\mathcal{D}_\mu(M)$ can be approximated by a corresponding index form in T_xM. This construction is not possible on a two-dimensional manifold. Thus we can find monoconjugate points on $\mathcal{D}_\mu(M^3)$ just by solving a simple ordinary differential equation in T_xM, for any point $x \in M$:

$$\frac{d}{dt}\Big(\Lambda(t,x)\frac{du}{dt}\Big) + \operatorname{curl} X_o(x) \times \frac{du}{dt} = 0, \tag{1.1}$$

where $\Lambda(t,x) = D\eta(t,x)^\star D\eta(t,x)$ is the metric pullback. If for some x in the interior of M, equation (1.1) has a solution $u(t)$ vanishing at times $t = a$ and $t = b$, then the geodesic η has a Jacobi field vanishing at the endpoints of any larger interval.

Although Theorem 3.1 applies only in three (or possibly higher) dimensions, it has a sort of converse that is true in dimension two or higher and is easily proven in Proposition 3.6. As one application, we find that harmonic vector fields on M (if the topology of M allows them) generate geodesics in $\mathcal{D}_\mu(M)$ that have no monoconjugate pairs, and are thus infinitesimally minimizing on their entire length.

The results of Theorem 3.1 and Proposition 3.6 are quite reminiscent of the results of Friedlander and Vishik [FV], though the method of proof is very different. They found ordinary differential equations such that exponential growth of their solutions implies exponential growth of linearized Euler perturbations, and hence of Jacobi fields along geodesics. Our result on conjugate points is loosely related to positive curvature on $\mathcal{D}_\mu$, while their result on exponential growth of Jacobi fields is loosely related to negative curvature on $\mathcal{D}_\mu$. But although the connection between curvature and Jacobi fields is subtle (as discussed in [P1]), both results show that the most important features of Jacobi fields in $\mathcal{D}_\mu(M^3)$ are determined by certain ordinary differential equations at an arbitrary point.

The criterion of Theorem 3.1 yields a very simple condition for conjugate points, which is easiest to apply when we are dealing with a steady solution X of the 3-D Euler equations. We find that for any steady solution X with a certain type of fixed point at some x (for example, an elliptic fixed point), there must be a monoconjugate point somewhere along the geodesic, and we can compute its location as the solution of a certain algebraic equation in terms of the stress tensor and the vorticity at x. This is a purely three-dimensional phenomenon: in two dimensions, there are many steady flows that have elliptic fixed points but do not have any conjugate points along the corresponding geodesic, because the curvature operator is nonpositive in all directions. See [P2] for details.

The technique we use suggests that the first conjugate point along a geodesic in $\mathcal{D}_\mu(M^3)$ is typically of the same form as the one found on $D^2 \times S^1$ in [EMP]: an epiconjugate point that is a limit of a decreasing sequence of monoconjugate points. We do not prove this, but it is a reasonable conjecture. This would imply that the pathological behavior of conjugate points is quite typical for diffeomorphism groups of 3-dimensional manifolds.

In Section 4, we apply the technique of Theorem 3.1 to study the blowup problem for the 3-D Euler equations. Our starting point is the famous criterion of Beale, Kato, and Majda [BKM], which states that a C^∞ initial condition X_o leads to a solution $X(t)$ that fails to exist in C^∞ after time T if and only if the quantity $\int_0^T \sup_{x\in M}\lvert\operatorname{curl} X(t,x)\rvert\, dt$

diverges to infinity. We conjecture that in fact we have the stronger condition that $\int_0^T \big| \operatorname{curl} X\big(t, \eta(t,x)\big)\big| \, dt = \infty$ for some x in the interior of M.

By computing a certain trace of the index form on T_xM, we are motivated to pose the Weak Conjugate Conjecture 4.2. This conjecture, a condition on the stress tensor and the vorcicity along the path $\eta(t,x)$ of maximum vorticity, implies via Theorem 4.3 that along a geodesic which ends at time T, there are monoconjugate pairs arbitrarily close to T. Thus if the Weak Conjugate Conjecture holds, the geodesic will fail to be minimizing on small intervals arbitrarily close to the blowup time T.

We present some simple conditions on the velocity field which imply the Weak Conjugate Conjecture, and list these in the Strong Conjugate Conjecture 4.5. As evidence for the Strong Conjugate Conjecture, we cite numerical studies by Bell and Marcus [BM], Kerr [K], Ohkitani and Kishiba [OK], and Pumir and Siggia [PS]. The well-known conjectured properties of solutions that blow up, which together imply the Strong Conjugate Conjecture, are that the maximum vorticity vector aligns with an eigenvector of the stress tensor; that the maximum vorticity magnitude diverges as $t \to T$ like $1/(T-t)$; and that the Laplacian of the pressure function remains positive along the path of maximum vorticity.

The main consequence of the Conjugate Conjectures and Theorem 4.3 is that a C^∞ solution that does not exist past a time T cannot be extended to a weak solution that extends past T, if we are in a class of weak solutions that locally minimizes the energy integral. The reason is that the conjugate points prevent minimization arbitrarily close to T, so that a proposed solution cannot minimize the energy integral in any neighborhood of the blowup time T. If it were known that there is a weak class in which global solutions that locally minimize energy always exist (Brenier's space of generalized flows [B1] is a natural candidate), then we would obtain a contradiction which would yield a true global existence theorem in C^∞.

Of course this program is rather speculative, but the evidence from numerical studies suggests it may be a useful avenue to pursue.

2. Background

In this section, we briefly review the geometry of the volume-preserving diffeomorphism group $\mathcal{D}_\mu(M)$. Many of the formulas provided here for covariant derivatives, curvature operators, and the index form were derived in [M1], [P1], and [EMP]. We will confine ourselves to the C^∞ case, although for some technical proofs it is more convenient to use the Sobolev H^s spaces. See Ebin and Marsden [EMa] for the precise constructions.

The space of volume-preserving diffeomorphisms $\mathcal{D}_\mu(M)$ of a Riemannian manifold M (possibly with boundary ∂M) consists of those C^∞ diffeomorphisms η satisfying $\eta^*\mu = \mu$, where μ is the Riemannian volume form. This space has the structure of a Fréchet manifold. Its tangent spaces $T_\eta\mathcal{D}_\mu(M)$ consist of elements of the form $X \circ \eta$, where X is a vector field on M that is divergence-free and tangent to the boundary. The L^2 Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{D}_\mu(M)$ is defined in terms of the metric $\langle \cdot, \cdot \rangle$ on

M by the formula

$$\langle\langle U\circ\eta, V\circ\eta\rangle\rangle = \int_M \langle U, V\rangle \circ\eta\, \mu. \tag{2.2}$$

$\mathcal{D}_\mu(M)$ also has a Lie group structure, where the group operator is composition. The differentials of the translation operators at the identity are

$$dL_\eta(X) = D\eta(X) \qquad \text{and} \qquad dR_\eta(X) = X\circ\eta. \tag{2.3}$$

By the change of variables formula for integrals, and the fact that each η is volume-preserving, we see that the metric (2.2) is right-invariant. It is not, however, left-invariant. In some sense, then, all the geometric information about $\mathcal{D}_\mu(M)$ is contained in the left-translations.

To compute covariant derivatives in the metric (2.2), we use the Weyl decomposition of vector fields. This decomposition allows us to write any vector field X on a manifold M as

$$X = U + \nabla f,$$

where U is divergence-free and tangent to ∂M. We construct this decomposition by solving the Neumann problem

$$\Delta f = \operatorname{div} X, \qquad \langle \nabla f, \nu\rangle_{\partial M} = \langle X, \nu\rangle_{\partial M} \tag{2.4}$$

for f, then defining $U = X - \nabla f$. (Here ν is the unit normal on ∂M.) This decomposition is orthogonal in the L^2 metric (2.2). We will denote the orthogonal projections by

$$P(X) = U \qquad \text{and} \qquad Q(X) = \nabla f. \tag{2.5}$$

By right-invariance of the metric, the orthogonal projections in $T_\eta\mathcal{D}_\mu(M)$ are given by

$$P_\eta = dR_\eta \circ P \circ dR_{\eta^{-1}} \qquad \text{and} \qquad Q_\eta = dR_\eta \circ Q \circ dR_{\eta^{-1}}. \tag{2.6}$$

Now we consider covariant derivatives.

Proposition 2.1. *Suppose $\eta(t)$ is a curve in $\mathcal{D}_\mu(M)$, and $J(t)$ is a vector field along $\eta(t)$. Let $X(t)$ be the Eulerian velocity field of η, defined by the formula*

$$X(t) = dR_{\eta(t)^{-1}}\left(\frac{d\eta}{dt}\right) = \frac{\partial\eta}{\partial t}\circ\eta(t)^{-1}. \tag{2.7}$$

There are three ways to compute the covariant derivative $\frac{\widetilde{\mathbf{D}}J}{dt}$ along η.

- *If we right-translate back to the identity to obtain $Y(t) = dR_{\eta(t)^{-1}}\big(J(t)\big)$, we get*

$$\frac{\widetilde{\mathbf{D}}J}{dt} = dR_{\eta(t)}\left(\frac{\partial Y}{\partial t} + P\big(\nabla_{X(t)}Y(t)\big)\right). \tag{2.8}$$

- *If we left-translate back to the identity to obtain $U(t) = dL_{\eta(t)^{-1}}\big(J(t)\big)$, we get*

$$\frac{\widetilde{\mathbf{D}}J}{dt} = dL_{\eta(t)}\left(\frac{\partial U}{\partial t}\right) + P_{\eta(t)}\big(\nabla_{J(t)}X(t)\big). \tag{2.9}$$

- *If we consider that for each fixed x, $\eta(t,x)$ is a curve in M and $J(t,x)$ is a vector field along that curve, we get*

$$\frac{\widetilde{\mathbf{D}}J}{dt} = P_{\eta(t)}\left(\frac{DJ}{dt}\right). \tag{2.10}$$

Here $\frac{DJ}{dt}(t,x)$ is the covariant derivative as computed in M along each curve $\eta(t,x)$.

Proof. Formula (2.8) was derived by Misiołek [M1].

Formula (2.9) can be derived from formula (2.8), using the fact that $Y(t) = \eta(t)_* U(t)$. From the definition of the Lie derivative (see [Sp]), we have

$$\frac{\partial Y}{\partial t} = \eta(t)_* \frac{\partial U}{\partial t} - [X(t), Y(t)], \tag{2.11}$$

so that

$$\frac{\partial Y}{\partial t} + \nabla_{X(t)}Y(t) = \eta(t)_* \frac{\partial U}{\partial t} + \nabla_{Y(t)}X(t). \tag{2.12}$$

Right-translating to $\eta(t)$ and applying the projection $P_{\eta(t)}$ to both sides, we see that (2.8) and (2.9) are equivalent.

Formula (2.10) also follows from (2.8), using the formula

$$\frac{DJ}{dt}(t,x) = \frac{\partial Y}{\partial t}\big(t, \eta(t,x)\big) + \nabla_{X(t,\eta(t,x))}Y. \tag{2.13}$$

Applying the translated projection $P_{\eta(t)}$ to both sides of (2.13), we see that (2.8) and (2.10) are equivalent. □

The geodesic equation on $\mathcal{D}_\mu(M)$ is

$$\frac{\widetilde{\mathbf{D}}}{dt}\frac{d\eta}{dt} = 0.$$

Using equation (2.8), the geodesic equation becomes, in terms of the Eulerian velocity field $X(t)$ defined by (2.7), the Euler equation of ideal incompressible flow:

$$\frac{\partial X}{\partial t} + \nabla_{X(t)}X(t) = -\nabla p(t). \tag{2.14}$$

The pressure function $p(t)$ is written with a negative sign by convention, and comes from solving the equation (2.4):

$$\nabla p(t) = -Q\big(\nabla_{X(t)}X(t)\big). \tag{2.15}$$

A common simplification is to write the pressure as the solution of the equation

$$\Delta p = -\operatorname{div}(\nabla_X X) = -\operatorname{Ric}(X) - \frac{1}{4}\operatorname{Tr}(\operatorname{Def} X)^2 + \frac{1}{2}|\operatorname{curl} X|^2, \tag{2.16}$$

where the deformation (stress) tensor is defined as

$$\langle \operatorname{Def} X(U), V\rangle = \langle \nabla_U X, V\rangle + \langle \nabla_V X, U\rangle.$$

By formula (2.10), we could also think of the geodesic equation as a family of ordinary differential equations on the manifold:

$$\frac{D}{dt}\frac{d\eta}{dt}(t,x) = -\nabla p(t,x), \tag{2.17}$$

which is Newton's equation with a time-dependent potential $p(t,x)$. Of course, $p(t,x)$ is not given in advance, but determined by the fluid so as to preserve volume. This point of view will be useful later; we will see how one can consider the full Jacobi equation on $\mathcal{D}_\mu(M)$ by comparing it to the far simpler pointwise Jacobi equation (with potential p) on M. Although these equations are not the same (when one is considering perturbations in M, one does not have the volume-preserving constraint to complicate the formulas), they are quite closely related.

We can eliminate the pressure term in (2.14) by computing the curl of both sides. In three dimensions, we get the vorticity form of the Euler equation:

$$\frac{\partial}{\partial t}\operatorname{curl} X(t,x) + \big[X(t,x), \operatorname{curl} X(t,x)\big] = 0. \tag{2.18}$$

Equation (2.18) implies that the vorticity is transported by the flow: for every $x \in M$,

$$\operatorname{curl} X\big(t, \eta(t,x)\big) = D\eta(t,x)\big(\operatorname{curl} X_o(x)\big). \tag{2.19}$$

More generally, by lowering indices in equation (2.14) to get an equation for the 1-form $X^\flat$ and taking the differential, we obtain the equation

$$\frac{\partial}{\partial t} dX^\flat + \mathcal{L}_X dX^\flat = 0, \tag{2.20}$$

the solution of which is

$$dX^\flat(t) = \big(\eta(t)^{-1}\big)^* dX_o^\flat. \tag{2.21}$$

See for example [AK] for details.

There are several formulas for the Riemann curvature tensor $\widetilde{\mathbf{R}}$ on $\mathcal{D}_\mu(M)$, but the only one we'll need is the following: if U, V, and W are divergence-free and tangent to the boundary, then

$$\widetilde{\mathbf{R}}(U,V)W = P\Big(R(U,V)W + \nabla_V Q(\nabla_U W) - \nabla_U Q(\nabla_V W)\Big). \tag{2.22}$$

See for example [P1]. Since the metric is right-invariant, the curvature tensor is as well, and thus we can compute the curvature at any $\eta \in \mathcal{D}_\mu(M)$ using the same formula.

We are interested in Jacobi fields, which are defined as follows: if $\eta(t,s)$ is a family of curves in $\mathcal{D}_\mu(M)$ with $\eta(t) = \eta(t,0)$ a geodesic, then $J(t) = \frac{\partial \eta}{\partial s}\big|_{s=0}$ is a Jacobi field along $\eta(t)$. Jacobi fields satisfy the linearized geodesic equation

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J(t), \dot\eta(t)\big)\dot\eta(t) = 0. \tag{2.23}$$

Equation (2.23) is extremely unwieldy, not least because the formulas for both curvature and the covariant derivative involve nonlocal operators (specifically, the solution of the Neumann problem (2.4)). However, the Jacobi equation can be simplified substantially, and in fact decoupled into two first-order equations. This fact was first observed by Rouchon [Ro], and exploited by the author [P1] to obtain explicit Jacobi fields along certain geodesics of $\mathcal{D}_\mu(M)$. These simplifications result from the following equivalent expressions for the linearization of the Newton equation (2.17).

Proposition 2.2. *Consider a solution $X(t)$ of the Euler equation (2.14), with corresponding geodesic $\eta(t)$ in $\mathcal{D}_\mu(M)$. Let $J(t)$ be a vector field along $\eta(t)$. Then we have*

$$\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R(J,\dot\eta)\dot\eta = \Big(\frac{\partial Z}{\partial t} + \nabla_X Z + \nabla_Z X\Big)\circ\eta, \tag{2.24}$$

where $Z = \partial_t Y + [X,Y]$ and $J = Y\circ\eta$.

We can also write

$$\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R(J,\dot\eta)\dot\eta = (D\eta^{-1})^\star \left(\frac{\partial}{\partial t}\big(\Lambda V\big) + \big(\imath_V dX_o^\flat\big)^\sharp\right), \tag{2.25}$$

where $V = \partial_t U$ and $J = D\eta(U)$, and $(D\eta^{-1})^\star$ is the pointwise metric adjoint of $D\eta^{-1}$.

Proof. To obtain equation (2.24), we start with

$$\frac{DJ}{dt} = (\partial_t Y + \nabla_X Y)\circ\eta = (Z + \nabla_Y X)\circ\eta,$$

a consequence of (2.13). Using the Euler equation (2.14), we have

$$\begin{aligned}
\Big(\frac{D^2 J}{dt^2} + \nabla_J\nabla p + R(J,\dot\eta)\dot\eta\Big)\circ\eta^{-1} &= \frac{D}{dt}\big((Z+\nabla_Y X)\circ\eta\big)\circ\eta^{-1} + \nabla_Y\nabla p + R(Y,X)X\\
&= \partial_t(Z+\nabla_Y X) + \nabla_X(Z+\nabla_Y X) - \nabla_Y(\partial_t X + \nabla_X X) + R(Y,X)X\\
&= \partial_t Z + \nabla_{\partial_t Y}X + \nabla_Y(\partial_t X) + \nabla_X Z + \nabla_X\nabla_Y X - \nabla_Y(\partial_t X) - \nabla_Y\nabla_X X\\
&\qquad + \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X\\
&= \partial_t Z + \nabla_Z X + \nabla_X Z.
\end{aligned}$$

To derive equation (2.25), we first recall that $Y = \eta_* U$, so that by equation (2.11), we have $\eta_*\partial_t U = \partial_t Y + [X,Y]$. Thus $Z = \eta_* V$. For convenience, define $L = Z\circ\eta = D\eta(V)$.

Then by equations (2.24) and (2.13), our goal becomes to prove that

$$\frac{DL}{dt} + \nabla_L X = (D\eta^{-1})^\star \left(\frac{\partial}{\partial t}\big(\Lambda V\big) + \big(\imath_V dX_o^\flat\big)^\sharp\right). \tag{2.26}$$

This equation involves no space derivatives, so we can consider it as an equation along the fixed curve $\eta(t,x)$ for each particular $x\in M$.

So for some fixed x, pick an arbitrary vector $w_o \in T_x M$. Then we can compute

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V)\Big\rangle = \frac{d}{dt}\langle w_o, \Lambda V\rangle = \frac{d}{dt}\langle D\eta(w_o), L\rangle. \tag{2.27}$$

By equations (2.12) and (2.13), we have the formula

$$\frac{D}{dt}\big(D\eta(w_o)\big) = \nabla_{D\eta(w_o)}X.$$

Thus (2.27) yields

$$\begin{aligned}\Big\langle w_o, \frac{d}{dt}(\Lambda V)\Big\rangle &= \big\langle \nabla_{D\eta(w_o)}X, L\big\rangle + \Big\langle D\eta(w_o), \frac{DL}{dt}\Big\rangle \\ &= \big\langle \nabla_{D\eta(w_o)}X, L\big\rangle - \big\langle D\eta(w_o), \nabla_L X\big\rangle + \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X\Big\rangle.\end{aligned}$$

Using the general formula

$$\langle \nabla_A X, B\rangle - \langle \nabla_B X, A\rangle = dX^\flat(A, B),$$

we can write

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V)\Big\rangle = \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X\Big\rangle - dX^\flat\big(D\eta(V), D\eta(w_o)\big).$$

Now since the vorticity 2-form is transported by the flow, equation (2.21) yields

$$dX^\flat\big(D\eta(V), D\eta(w_o)\big) = dX_o^\flat(V, w_o) = \langle \big(\imath_V dX_o^\flat\big)^\sharp, w_o\rangle.$$

Thus we finally get

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V)\Big\rangle = \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X\Big\rangle - \langle \big(\imath_V dX_o^\flat\big)^\sharp, w_o\rangle,$$

and since this is true for any $w_o \in T_x M$, we have the equation

$$D\eta^\star\Big(\frac{DL}{dt} + \nabla_L X\Big) = \frac{d}{dt}(\Lambda V) + \big(\imath_V dX_o^\flat\big)^\sharp,$$

which yields (2.26) and hence (2.25). □

Remark 2.3. Using Proposition 2.2, we can rewrite the ordinary differential equation of Friedlander and Vishik [FV] in a manner analogous to (1.1). For a steady solution X of the Euler equation on $\mathbb{T}^3$ and for some $x \in \mathbb{T}^3$, their equation becomes, using (2.26),

$$\frac{d}{dt}\Big(\Lambda(t,x)v(t)\Big) + \operatorname{curl} X_o(x) \times v(t) = 2\frac{\langle w_o, D\eta(t,x)^{-1}\nabla_{D\eta(t,x)v(t)}X\rangle}{\langle w_o, \Lambda(t,x)^{-1}w_o\rangle}w_o, \tag{2.28}$$

with w_o some constant vector orthogonal to $v(0)$. If there is a solution $v(t)$ of (2.28) such that $\langle v(t), \Lambda(t,x)v(t)\rangle$ grows exponentially in time, then there is a Jacobi field along the corresponding geodesic in $\mathcal{D}_\mu(\mathbb{T}^3)$ that grows exponentially in the L^2 metric. In [FV], a number of conditions are given on X which yield exponential instability; conceivably, the modified form (2.28) may suggest some new ones. However, we leave this direction of research aside for now.

From Proposition 2.2, we can derive the following, which will be useful later.

Proposition 2.4. *If M is a three-dimensional manifold and X and η are as given in Proposition 2.2, then for any $x \in M$, the vorticity field $J(t) = \operatorname{curl} X\big(t, \eta(t,x)\big)$ is a solution of the linearized Newton equation*

$$\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R(J, \dot\eta)\dot\eta = 0 \tag{2.29}$$

along the curve $\eta(t,x)$.

Proof. Since the vorticity is transported, by (2.19), $J = \operatorname{curl} X \circ \eta$ has $U = dL_{\eta^{-1}}(J) = \operatorname{curl} X_o$ and $V = \partial_t U = 0$. So by equation (2.25), we know (2.29) is satisfied. □

The following proposition shows how the Jacobi equation simplifies under either left- or right-translations.

Proposition 2.5. *If η is a geodesic in $\mathcal{D}_\mu(M)$ and X is its Eulerian velocity field defined by (2.7), then the Jacobi operator in (2.23) can be written in two ways:*

- *in terms of the right-translation, with $Y = dR_{\eta^{-1}}(J)$, as*

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J, \dot\eta\big)\dot\eta = dR_\eta \left(\frac{\partial Z}{\partial t} + P\Big(\nabla_Z X + \nabla_X Z\Big)\right), \tag{2.30}$$

  where

$$Z = \partial_t Y + [X, Y]. \tag{2.31}$$

- *in terms of Jacobi operators along the paths $\eta(t,x)$, as*

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J, \dot\eta\big)\dot\eta = P_\eta \left(\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R\big(J, \dot\eta\big)\dot\eta\right) \tag{2.32}$$

- *in terms of the left-translation, with $U = dL_{\eta^{-1}}\big(J\big)$, as*

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J, \dot\eta\big)\dot\eta = (dL_{\eta^{-1}})^\star \left(\frac{\partial}{\partial t} P\Big(\Lambda \frac{\partial U}{\partial t}\Big) + K_{X_o}\Big(\frac{\partial U}{\partial t}\Big)\right), \tag{2.33}$$

  where the operator $K_{X_o} \colon T_{id}\mathcal{D}_\mu(M) \to T_{id}\mathcal{D}_\mu(M)$ is defined by

$$K_{X_o}(W) = P\big(\imath_W dX_o^\flat\big)^\sharp, \tag{2.34}$$

  with $\Lambda = D\eta^\star D\eta$ being the metric pullback, X_o being the initial velocity field, and

$$(dL_{\eta^{-1}})^\star = dR_\eta \circ P \circ (D\eta^{-1})^\star \circ dR_{\eta^{-1}}$$

  being the L^2 adjoint of the operator $dL_{\eta^{-1}} \colon T_{id}\mathcal{D}_\mu \to T_{\eta^{-1}}\mathcal{D}_\mu$.

Proof. The right-translated Jacobi equation (2.30) was derived by Rouchon [Ro], by linearizing the geodesic equation (2.14) and (2.7) directly. The fact that (2.30) is equivalent to (2.23) can be seen directly, using formulas (2.8) and (2.22).

Equation (2.32) is an immediate consequence of equation (2.24). Equation (2.33) is a consequence of (2.25), along with the observation that $(dL_{\eta^{-1}})^\star(W) = (dL_{\eta^{-1}})^\star \circ P(W)$

for any vector field W. This observation follows from the fact that if g is a function on M, then

$$(dL_{\eta^{-1}})^\star(\nabla g) = dR_\eta \circ P \circ (D\eta^{-1})^\star \circ dR_{\eta^{-1}}(\nabla g) = dR_\eta \circ P\big(\nabla(g\circ\eta^{-1})\big) = 0.$$

□

The operator K_X defined by (2.34) is given in two dimensions by

$$K_{X_o}(W) = P\big((\operatorname{curl} X_o)\star W\big),$$

where $\star$ is the two-dimensional Hodge star operator that rotates vectors 90°. This operator is compact, as discussed in [EMP]. In three dimensions,

$$K_{X_o}(W) = P\big(\operatorname{curl} X_o \times W\big),$$

and this operator is not compact. The fact that this operator fails to be compact in three dimensions is the main reason Fredholmness of the exponential map fails in three dimensions, which is why conjugate points look so different between $\mathcal{D}_\mu(M^2)$ and $\mathcal{D}_\mu(M^3)$.

The main thing we are interested in for this paper is the index form along a geodesic $\eta(t)$ in $\mathcal{D}_\mu(M)$. In general this is defined for a Riemannian manifold as

$$I_{[a,b]}\big(J(t),J(t)\big) = \int_a^b \left\langle\!\!\left\langle \frac{\widetilde{\mathbf{D}}J}{dt}, \frac{\widetilde{\mathbf{D}}J}{dt} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \widetilde{\mathbf{R}}\Big(J(t), \frac{d\eta}{dt}\Big)\frac{d\eta}{dt}, J(t) \right\rangle\!\!\right\rangle \, dt. \tag{2.35}$$

The index form represents the second derivative of the energy functional

$$E(s) = \frac{1}{2}\int_a^b \left\langle\!\!\left\langle \frac{\partial\eta(t,s)}{\partial t}, \frac{\partial\eta(t,s)}{\partial t} \right\rangle\!\!\right\rangle dt.$$

If $\eta(t,s)$ is a family of curves in $\mathcal{D}_\mu(M)$, such that $\eta(t,0)$ is a geodesic, with $\eta(a,s)$ and $\eta(b,s)$ constant in s, then $E'(0) = 0$ and

$$E''(0) = I_{[a,b]}\Big(\frac{\partial\eta}{\partial s}\Big|_{s=0}, \frac{\partial\eta}{\partial s}\Big|_{s=0}\Big).$$

So if the index form is negative for some vector field $J(t)$ vanishing at $t=a$ and $t=b$, then the geodesic is not minimizing on $[a,b]$. In addition, there must be a Jacobi field which vanishes at $t=a$ and $t=c$ for some $c\in(a,b)$.

We will derive several alternative formulas for the index form (2.35), which will form the basis for the rest of the paper.

Proposition 2.6. *If $\eta(t)$ is a geodesic in $\mathcal{D}_\mu(M)$ and $J(t)$ is a smooth vector field along $\eta(t)$ vanishing at $t=a$ and $t=b$, then the index form $I_{[a,b]}\big(J(t),J(t)\big)$ may be written in either of the following forms:*

- *in terms of covariant derivatives on M as*

$$I_{[a,b]}\big(J(t),J(t)\big) = \int_a^b \int_M \Big\langle \frac{DJ}{dt}(t,x), \frac{DJ}{dt}(t,x)\Big\rangle - \Big\langle \nabla_{J(t,x)}\nabla p + R\big(J(t,x),\dot\eta(t,x)\big)\dot\eta(t,x), J(t,x)\Big\rangle\, \mu(x)\, dt, \tag{2.36}$$

- *or in terms of the left-translation $U(t) = dL_{\eta(t)^{-1}}\big(J(t)\big)$ as*

$$I_{[a,b]}\big(J(t),J(t)\big) = \int_a^b \int_M \Big\langle \Lambda(t,x)\frac{\partial U}{\partial t}(t,x), \frac{\partial U}{\partial t}(t,x)\Big\rangle + dX_o(x)^\flat\Big(U(t,x), \frac{\partial U}{\partial t}(t,x)\Big)\, \mu(x)\, dt, \tag{2.37}$$

where $\Lambda(t,x) = D\eta(t,x)^\star D\eta(t,x)$ is the metric pullback and $\operatorname{curl} X_o$ *is the initial vorticity.*

Proof. Both of these formulas follow immediately from Proposition 2.5, after integrating the index form (2.35) by parts to obtain

$$I_{[a,b]}\big(J(t),J(t)\big) = -\int_a^b \Big\langle\!\Big\langle \frac{\widetilde{\mathbf{D}}^2 J}{dt} + \widetilde{\mathbf{R}}\Big(J(t), \frac{d\eta}{dt}\Big)\frac{d\eta}{dt}, J(t)\Big\rangle\!\Big\rangle\, dt.$$

□

What is remarkable about both of the formulas (2.36) and (2.37) is that they involve only local computations; it is not necessary to solve the Neumann problem (2.4) to compute the index form. The index form is virtually the *only* object in the geometry of $\mathcal{D}_\mu$ that can be computed so easily, and it is this fact which helps so much to understand conjugate points on $\mathcal{D}_\mu$, despite our very incomplete understanding of the curvature on $\mathcal{D}_\mu$.

3. The local criterion

Theorem 3.1. *Let M be a 3-dimensional compact manifold (possibly with boundary). Let $\eta\colon [0,T) \to \mathcal{D}_\mu(M)$ be a geodesic curve in the diffeomorphism group with $\eta(0) = id$. (Here T is the maximal time of existence, which may be infinite.) Let $X(t) = \frac{d\eta}{dt}\circ\eta(t)^{-1}$ be the velocity field, with $X(0) = X_o$.*

If for some point x in the interior of M, the ordinary differential equation

$$\frac{d}{dt}\left(\Lambda(t,x)\,\frac{du}{dt}\right) + \operatorname{curl} X_o(x) \times \frac{du}{dt} = 0 \tag{3.38}$$

has a nontrivial solution vanishing at $t = 0$ and $t = a$, then for any $\delta > 0$, there is a $b \in (0, a+\delta)$ such that $\eta(b)$ is monoconjugate to id along η.

Proof. Clearly $\operatorname{curl} X_o(x)$ is not zero; if it were, we could not have a nontrivial solution vanishing at two points, since $\Lambda(t,x)$ is positive definite.

So set up an oriented orthonormal basis $\{e_1, e_2, e_3\}$ at T_xM, such that $\operatorname{curl} X_o(x) = \omega_o e_3$, with $\omega_o > 0$. Choose Riemannian normal coordinates (x_1, x_2, x_3) such that at x, $\partial_{x_1} = e_1$, $\partial_{x_2} = e_2$, and $\partial_{x_3} = e_3$.

Let $h\colon \mathbb{R} \to [-1, 1]$ be a C^∞ function which vanishes identically outside $[-1, 1]$.

For a small $\varepsilon > 0$, let us define three vector fields

$$\begin{aligned}
A_1 &= \varepsilon^4 h'\Big(\frac{x_1}{\varepsilon}\Big) h\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_1} - \varepsilon^3 h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_2},\\
A_2 &= \varepsilon^3 h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_1} + \varepsilon^4 h'\Big(\frac{x_1}{\varepsilon}\Big) h\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_2},\\
A_3 &= -\frac{\varepsilon^3}{2} h'\Big(\frac{x_1}{\varepsilon^2}\Big) h\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\, \partial_{x_1} + \frac{\varepsilon^2}{2} h\Big(\frac{x_1}{\varepsilon^2}\Big) h'\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\, \partial_{x_2}.
\end{aligned}$$

Now we specify divergence-free vector fields E_1, E_2, and E_3 by the formulas $E_j = \operatorname{curl} A_j$. Since we are working in Riemannian normal coordinates, we can compute these curls to order $O(\varepsilon)$ just using the Euclidean formulas, and we obtain:

$$\begin{aligned}
E_1 &= h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h'\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_1} + O(\varepsilon) \text{ on } [-\varepsilon, \varepsilon] \times [-\varepsilon^2, \varepsilon^2] \times [-\varepsilon^3, \varepsilon^3],\\
E_2 &= h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h'\Big(\frac{x_3}{\varepsilon^3}\Big)\, \partial_{x_2} + O(\varepsilon) \text{ on } [-\varepsilon, \varepsilon] \times [-\varepsilon^2, \varepsilon^2] \times [-\varepsilon^3, \varepsilon^3],\\
E_3 &= h'\Big(\frac{x_1}{\varepsilon^2}\Big) h'\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\, \partial_{x_3} + O(\varepsilon) \text{ on } [-\varepsilon^2, \varepsilon^2] \times [-\varepsilon^3, \varepsilon^3] \times [-\varepsilon, \varepsilon].
\end{aligned}$$

These vector fields are chosen so that, roughly speaking, E_i is nearly parallel to e_i near x, to lowest order. More precisely, we can check that the following formulas hold for the L^2 inner products:

$$\begin{gathered}
\langle\langle E_1, E_1\rangle\rangle = \langle\langle E_2, E_2\rangle\rangle = \langle\langle E_3, E_3\rangle\rangle = \Gamma\varepsilon^6 + O(\varepsilon^7)\\
\langle\langle E_1, E_2\rangle\rangle = \langle\langle E_1, E_3\rangle\rangle = \langle\langle E_2, E_3\rangle\rangle = O(\varepsilon^7)\\
\langle\langle \partial_{x_3} \times E_1, E_2\rangle\rangle = \Gamma\varepsilon^6 + O(\varepsilon^7), \qquad \langle\langle \partial_{x_3} \times E_2, E_1\rangle\rangle = -\Gamma\varepsilon^6 + O(\varepsilon^7)\\
\langle\langle \partial_{x_3} \times E_1, E_3\rangle\rangle = \langle\langle \partial_{x_3} \times E_2, E_3\rangle\rangle = O(\varepsilon^7),
\end{gathered}$$

where the constant Γ is defined by

$$\Gamma = \left(\int_{-1}^{1} h(\sigma)^2\, d\sigma\right) \left(\int_{-1}^{1} h'(\sigma)^2\, d\sigma\right)^2.$$

Now since $u(t)$ is a solution of equation (3.38) vanishing at $t = 0$ and $t = a$, we have that for any $\delta > 0$, there is a vector function $\tilde{u}(t)$ vanishing at $t = 0$ and $t = a + \delta$ such that

$$i_{a+\delta}(\tilde{u}, \tilde{u}) \equiv \int_0^{a+\delta} \langle \Lambda(t, x)\tilde{u}'(t), \tilde{u}'(t)\rangle + \langle \operatorname{curl} X_o(x) \times \tilde{u}(t), \tilde{u}'(t)\rangle\, dt < 0.$$

(The construction is the same as that for Jacobi fields in finite-dimensional Riemannian geometry, or more generally for index forms of second-order self-adjoint equations. See for example Reid [Re].)

If $\tilde{u}(t) = u^1(t)e_1 + u^2(t)e_2 + u^3(t)e_3$, then define $\tilde{U}(t) = u^1(t)E_1 + u^2(t)E_2 + u^3(t)E_3$. For y in the support of $\tilde{U}$, we can approximate $\Lambda(t,y) = \Lambda(t,x) + O(\varepsilon)$ and $\operatorname{curl} X_o(y) = \operatorname{curl} X_o(x) + O(\varepsilon)$. Therefore, we have

$$\begin{aligned} I_{a+\delta}(\tilde{U},\tilde{U}) &= \int_0^{a+\delta} \langle\langle \Lambda(t)\partial_t\tilde{U}, \partial_t\tilde{U}\rangle\rangle + \langle\langle \operatorname{curl} X_o \times \tilde{U}(t), \partial_t\tilde{U}\rangle\rangle\, dt \\ &= \int_0^{a+\delta} \langle\langle \Lambda(t)\partial_t\tilde{U}, \partial_t\tilde{U}\rangle\rangle + \omega_o\langle\langle \partial_{x_3} \times \tilde{U}, \partial_t\tilde{U}\rangle\rangle\, dt + O(\varepsilon^7) \\ &= \Gamma\varepsilon^6 i_{a+\delta}(\tilde{u},\tilde{u}) + O(\varepsilon^7), \end{aligned}$$

and choosing ε sufficiently small, we can make this quantity negative.

Since the index form is negative for some divergence free vector field on the interval $[0, a+\delta]$, there must be a Jacobi field along η vanishing at $t=0$ and $t=b$ for some $b < a+\delta$. So $\eta(b)$ is monoconjugate to $\eta(0)$ along η, as desired. $\square$

Remark 3.2. The main point is that for any particular vector $u \in T_xM$, we can construct a divergence-free vector field U such that $U \approx u$ near p and $P(\operatorname{curl} X_o \times U) \approx \operatorname{curl} X_o \times u$ near x. We can do this only in three (or possibly higher) dimensions. In two dimensions, the index form takes the form

$$I_{[0,a]}(U,U) = \int_0^a \langle\langle \Lambda(t)\partial_t U, \partial_t U\rangle\rangle + \langle\langle (\operatorname{curl} X_o) \star U, \partial_t U\rangle\rangle\, dt,$$

where $\star$ is the Hodge star operator. If U is any divergence-free vector field with support in a disc, then $\star U$ is a gradient, and thus to lowest order, $(\operatorname{curl} X_o) \star U$ is also a gradient.

Since the gradients are orthogonal to the divergence-free vector fields, the second term in the index form vanishes to lowest order; thus the index form is positive definite to lowest order. We conclude that there is no local criterion that can be used to find conjugate points along two-dimensional fluid flows: conjugate points on $\mathcal{D}_\mu(M^2)$ are an essentially global phenomenon. In three (and possibly higher) dimensions, conjugate points are essentially a local phenomenon.

Remark 3.3. The result is sharp, in the sense that there may not be a monoconjugate point actually at $\eta(a)$. This is precisely what happens for one example where we can compute everything explicitly: uniform rotation with angular velocity 1 of the solid torus $D^2 \times S^1$. Ebin, Misiołek, and the author [EMP] computed explicitly the Jacobi fields along this flow in terms of curl eigenfields on the cylinder, and found that monoconjugate points occurred at a sequence of locations that decreased to π, but that $\eta(\pi)$ itself was not a monoconjugate point. For this example $\Lambda(t,x)$ is always the identity and $\operatorname{curl} X_o \equiv 2\,\partial_{x_3}$, so that the equation (3.38) becomes $u''(t) + 2\,\partial_{x_3} \times u'(t) = 0$. With $u(0) = 0$, the solutions are

$$u(t) = \frac{1}{2}\begin{pmatrix} 1-\cos 2t & -\sin 2t & 0 \\ \sin 2t & 1-\cos 2t & 0 \\ 0 & 0 & t \end{pmatrix} u'(0),$$

and choosing $u'(0)$ orthogonal to ∂_{x_3}, we see that the first vanishing point is $a = \pi$.

If a is not actually a monoconjugate location, then there must be a sequence of monoconjugate locations descending to a. As a result, the seemingly pathological behavior described for uniform rotation of a solid torus in [EMP] is actually quite typical on $\mathcal{D}_\mu(M^3)$.

Theorem 3.1 is easiest to apply if $X = X_o$ is a steady solution of the Euler equation, with a fixed point x.

Theorem 3.4. *Suppose X is a steady solution of the Euler equation $P(\nabla_X X) = 0$ on a 3-manifold M, and x is a fixed point of X in the interior of M. Then there is an oriented orthonormal basis $\{e_1, e_2, e_3\}$ such that in this basis,* $\operatorname{Def} X(x) = \left(\begin{smallmatrix} 2\lambda & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ *and* $\operatorname{curl} X(x) = \omega_o e_3$*, with $\omega_o > 0$. The equation* (3.38) *has a nontrivial solution vanishing at $t = 0$ and some $t = a > 0$ if and only if $|\lambda| < \omega_o$.*

Proof. Obviously if $\operatorname{curl} X(x) = 0$, the solution of equation (3.38) with $u(0) = 0$ is

$$u(t) = \left(\int_0^t \Lambda(\tau, x)^{-1} \, d\tau \right) u'(0),$$

which vanishes only at $t = 0$, since the solution operator is positive-definite. Thus we know $\operatorname{curl} X(x) \neq 0$ is a necessary condition to obtain a solution.

Since X is assumed to be a steady solution of the Euler equation, we know by (2.18) that $[X, \operatorname{curl} X] = 0$ everywhere, and in particular at x. Thus $\nabla_{\operatorname{curl} X(x)} X = \nabla_{X(x)} \operatorname{curl} X = 0$ since $X(x) = 0$. Now for any $Y \in T(x)M$, we can write $\nabla_Y X = \frac{1}{2} \operatorname{Def} X(x)(Y) + \frac{1}{2} \operatorname{curl} X(x) \times Y$. Therefore we know that

$$0 = \nabla_{\operatorname{curl} X(x)} X = \frac{1}{2} \operatorname{Def} X(x)(\operatorname{curl} X(x)).$$

Thus $\operatorname{curl} X(x)$ is an eigenvector of the self-adjoint operator $\operatorname{Def} X(x)$ with eigenvalue 0.

Since $\operatorname{div} X = 0$, $\operatorname{Def} X(x)$ must have trace zero, and thus the other two real eigenvalues add to 0. So in an oriented, orthonormal basis of eigenvectors $\{e_1, e_2, e_3\}$ such that $\operatorname{curl} X(x) = \omega_o e_3$, we can write

$$\operatorname{Def} X(x) = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where λ is some real number.

Since x is a fixed point, the metric pullback $\Lambda(t, x)$ is given by $\Lambda(t, x) = e^{t \operatorname{Def} X(x)}$, and thus equation (3.38) takes the form

$$\frac{d}{dt} \left[\begin{pmatrix} e^{2\lambda t} & 0 & 0 \\ 0 & e^{-2\lambda t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{du}{dt} \right] + \omega_o \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{du}{dt} = 0. \tag{3.39}$$

Differentiating once and performing some algebra, we can obtain constant-coefficient equations for the components of u, and thus without much difficulty we can obtain the general solution with $u(0) = 0$ in the form $u(t) = \Phi(t)u'(0)$, with the solution operator $\Phi(t)$ given by

$$\Phi(t) = \frac{1}{\omega_o}\begin{pmatrix} \omega_o e^{\lambda t}S_\mu(t) & 1 - e^{\lambda t}C_\mu(t) + \lambda e^{\lambda t}S_\mu(t) & 0 \\ e^{-\lambda t}C_\mu(t) + \lambda e^{-\lambda t}S_\mu(t) - 1 & \omega_o e^{-\lambda t}S_\mu(t) & 0 \\ 0 & 0 & \omega_o t \end{pmatrix} \tag{3.40}$$

where $\mu = \omega_o^2 - \lambda^2$ and the functions $S_\mu(t)$ and $C_\mu(t)$ represent the solutions of $f''(t) = -\mu f(t)$ with initial conditions

$$S_\mu(0) = 0, \quad S_\mu'(0) = 1, \qquad C_\mu(0) = 1, \quad C_\mu'(0) = 0.$$

We will have $u(a) = 0$ for some $a > 0$ and some initial condition $u'(0)$ if and only if the determinant of the matrix appearing in (3.40) vanishes at $t = a$. Since the functions S_μ and C_μ satisfy the identity $C_\mu^2(t) + \mu S_\mu^2(t) = 1$, we can compute the determinant of this matrix as

$$\det \Phi(t) = \frac{2t}{\omega_o^2}\big[1 - \cosh \lambda t\, C_\mu(t) + \lambda \sinh \lambda t\, S_\mu(t)\big].$$

We now consider two cases. First, if $\mu \le 0$, then the derivative of the term inside the square brackets (using ${S_\mu}'(t) = C_\mu(t)$ and ${C_\mu}'(t) = -\mu S_\mu(t)$) is

$$\frac{d}{dt}\left[\frac{\omega_o^2}{2t}\det \Phi(t)\right] = \omega_o^2 \cosh \lambda t\, S_\mu(t). \tag{3.41}$$

Now if $\mu < 0$ then $S_\mu(t) = \frac{1}{\sqrt{|\mu|}}\sinh \sqrt{|\mu|}t$, and if $\mu = 0$ then $S_\mu(t) = t$. Either way, the function on the right side of (3.41) is always positive for $t > 0$, which means that $\frac{\omega_o^2}{2t}\det \Phi(t)$ is strictly increasing for $t > 0$, and thus that $\det \Phi(t)$ is also strictly increasing. Thus it starts at zero and is positive for all $t > 0$.

The other case is when $\mu > 0$. In this case, $S_\mu(t) = \frac{1}{\sqrt{\mu}}\sin \sqrt{\mu}t$ and $C_\mu(t) = \cos \sqrt{\mu}t$. If $\lambda = 0$ then we clearly have $\det \Phi(t) = 0$ at $t = 2\pi/\omega_o$. If $\lambda \neq 0$, then we can write

$$\det \Phi(t) = \frac{t}{\omega_o^2}\left[2 - \frac{\omega_o}{\sqrt{\mu}}e^{|\lambda| t}\cos\left(\sqrt{\mu}t + \phi\right) - \frac{\omega_o}{\sqrt{\mu}}e^{-|\lambda| t}\cos\left(\sqrt{\mu}t - \phi\right)\right],$$

where $\phi = \arctan\left(|\lambda|/\sqrt{\mu}\right)$. For t sufficiently large, the middle term in brackets will take on arbitrarily large positive and negative values, while the term on the right can be made arbitrarily small. Thus the sum must be equal to -2 for some $t = a$, which will make $\det \Phi(a) = 0$, as desired. Thus there will be some choice of $u'(0)$ that makes $u(a) = 0$. □

Remark 3.5. By formula (2.16), we have

$$\Delta p(x) + \mathrm{Ric}(X(x)) = -2\lambda^2 + \frac{1}{2}\omega^2.$$

Thus if $\Delta p + \mathrm{Ric}(X) \ge 0$ at the fixed point x, we will have $|\lambda| \le \frac{1}{2}|\omega|$, so that the hypothesis of Theorem 3.1 is automatically satisfied. The nonnegativity requirement for the pressure Laplacian is a theme that will recur in Section 4. This theorem shows that something akin to a nonnegativity requirement is necessary to obtain conjugate points.

We have a natural converse to Theorem 3.1, which works for any dimension $n \geq 2$.

Proposition 3.6. *Let M be any manifold with dimension $n \geq 2$, possibly with boundary. Suppose $\eta(t)$ is a geodesic in $\mathcal{D}_\mu(M)$ with $\eta(0) = id$, and let X be the Eulerian velocity field defined by $X = \frac{\partial\eta}{\partial t} \circ \eta^{-1}$.*

If there is a Jacobi field along η vanishing at $t = 0$ and $t = a > 0$, then for some x in the interior of M, there is a solution $u(t)$ of the ordinary differential equation

$$\frac{d}{dt}\left(\Lambda(t,x)\,\frac{du}{dt} + \big(\imath_{u(t)} dX_o^\flat(x)\big)^\sharp\right) = 0 \tag{3.42}$$

with $u(0) = u(a) = 0$.

Proof. Let $U(t)$ be the left translation of the Jacobi field vanishing at $t = 0$ and $t = a$. Then the index form $I_{[0,a]}(U,U)$ vanishes:

$$\int_0^a \langle\langle \Lambda(t)U_t(t), U_t(t)\rangle\rangle + \langle\langle \big(\imath_{U(t)} dX_o^\flat\big)^\sharp, U_t(t)\rangle\rangle\,dt = 0.$$

Thus, interchanging the order of integration, we know that

$$\int_M \int_0^a \langle \Lambda(t,x)U_t(t,x), U_t(t,x)\rangle + \langle \big(\imath_{U(t,x)} dX_o^\flat(x)\big)^\sharp, U_t(t,x)\rangle\,dt\,d\mu(x) = 0.$$

As a result, the function

$$i_a(x) = \int_0^a \langle \Lambda(t,x)U_t(t,x), U_t(t,x)\rangle + \langle \big(\imath_{U(t,x)} dX_o^\flat(x)\big)^\sharp, U_t(t,x)\rangle\,dt \tag{3.43}$$

must vanish for some x in the interior of M.

Now $i_a(x)$ is the index form of the self-adjoint system (3.42), and since the matrix $\Lambda(t,x)$ is always positive-definite, we can apply the Morse index theorem for systems to conclude that if $i_a(x) = 0$, then the vector $u(t) = U(t,x)$ must satisfy (3.42). See Reid [Re], Theorem V.8.1. □

We have an easy corollary that guarantees the nonexistence of monoconjugate points along certain flows.

Corollary 3.7. *Let M be a manifold of dimension at least two. If X is a harmonic vector field on M, i.e., one satisfying* $\operatorname{div} X \equiv 0$ *and $dX^\flat \equiv 0$, with X tangent to the boundary of M (if any), then X is a steady solution of the ideal Euler equation, and the corresponding geodesic in $\mathcal{D}_\mu(M)$ has no monoconjugate pairs.*

Proof. X is a steady Euler flow by equation (2.20). Since $dX^\flat \equiv 0$, we know the function $i_a(x)$ defined by (3.43) is positive-definite on M. So there cannot be any solution of (3.42) vanishing at two times. □

Remark 3.8. We conjecture the following property of the first conjugate point $\eta(a)$ (if any) along a geodesic in $\mathcal{D}_\mu(M^3)$. Either $\eta(a)$ is epiconjugate and not monoconjugate to $\eta(0)$, with a decreasing sequence of monoconjugate points converging to $\eta(a)$; or $\eta(a)$ is a monoconjugate point of infinite order. Either type implies a rather drastic failure of Fredholmness already at the first conjugate point. We will not try to prove this here, since our main concern is applying the conjugate point theorems above to the blowup problem; however we suspect the proof is not difficult.

4. Blowup

The celebrated paper of Beale, Kato, and Majda [BKM] showed that if a C^∞ solution of the 3-D Euler equation (2.14) fails to exist at a time T, then the following equation must be satisfied:

$$\int_0^T \sup_{x\in M} |\operatorname{curl} X(t,x)|\,dt = \infty. \tag{4.44}$$

It is generally believed, and has been observed in numerical simulations, that this blowup condition in fact occurs along a particular path, i.e., that for some x in the interior of M, we have

$$\int_0^T |\operatorname{curl} X\big(t,\eta(t,x)\big)|\,dt = \infty. \tag{4.45}$$

We will assume this stronger form of the Beale-Kato-Majda criterion, which to the author's knowledge has not yet been proven.

For the purpose of the next conjecture, we recall the definition of oscillatory differential equations from Swanson [Sw].

Definition 4.1. A differential equation for a scalar function $g(s)$ of the form

$$g''(s) + c(s)g(s) = 0,$$

with $c(s)$ some given function, is called *oscillatory* on $(0,\infty)$ if one (and hence every) solution $g(s)$ has infinitely many zeroes on $(0,\infty)$.

We pose the following conjecture. Its motivation will become clear in Theorem 4.3.

Conjecture 4.2 (Weak Conjugate Conjecture)**.** Suppose η is a geodesic in $\mathcal{D}_\mu(M^3)$ which exists on a time interval $[0,T)$. Let $X(t)$ be the velocity field, and let $\omega(t) = \operatorname{curl} X(t)$ be the vorticity field. Suppose that for some point x in the interior of M, we have the Beale-Kato-Majda blowup condition (4.45).

Define a time-dependent function by

$$\Psi(t) = -\frac{1}{4}\Big(\frac{|\omega\times \operatorname{Def} X(\omega)|}{|\omega|^3}\Big)^2 + \frac{1}{2}\,\frac{\Delta p + \operatorname{Ric}(X)}{|\omega|^2} + \frac{3}{4}\Big(\frac{1}{|\omega|^2}\,\frac{d|\omega|}{dt}\Big)^2, \tag{4.46}$$

where all quantities are evaluated at time t and position $\eta(t,x)$. Let $s = h(t) \equiv \int_0^t |\omega(\tau,x)|\,d\tau$. By (4.45), h maps $[0,T)$ bijectively to $[0,\infty)$.

The conjecture is that the ordinary differential equation

$$g''(s) + \Psi\big(h^{-1}(s)\big)\, g(s) = 0 \tag{4.47}$$

is oscillatory on $(0, \infty)$,

Theorem 4.3. *Suppose η is a geodesic in $\mathcal{D}_\mu(M^3)$ which exists on a time interval $[0, T)$. Let $X(t)$ be the velocity field, and let $\omega(t) = \operatorname{curl} X(t)$ be the vorticity field. Suppose that for some point x in the interior of M, we have the Beale-Kato-Majda blowup condition* (4.45).

If the Weak Conjugate Conjecture 4.2 holds, then for every $t_o \in (0, T)$ there is a time $t_1 \in (t_o, T)$ such that $\eta(t_1)$ is monoconjugate to $\eta(t_o)$ along η.

Proof. By Theorem 3.1 and Proposition 2.6, it is sufficient to show that there is a vector field $y(t)$ along $\eta(t, x)$ vanishing at times t_o and some $t_2 > t_o$, such that the index form is negative:

$$i_{[t_o,t_2]}(y, y) = \int_{t_o}^{t_2} \left\langle \frac{Dy}{dt}, \frac{Dy}{dt} \right\rangle - \big\langle \nabla_y \nabla p + R(y, \dot\eta)\dot\eta, y \big\rangle \, dt < 0. \tag{4.48}$$

Then our desired t_1 will lie in (t_o, t_2).

Our method will be to imitate the technique of the Bonnet-Myers theorem. This technique yields a quite different result here, since we are dealing with perturbations of the Newton equation $\frac{D}{dt}\frac{d\eta}{dt} = -\nabla p \circ \eta$ rather than the geodesic equation $\frac{D}{dt}\frac{d\eta}{dt} = 0$. One reason for the difference is that in the Riemannian case, the tangent vector is a solution of the Jacobi equation. In the Newtonian case, this is not true; however the vector field $\omega = \operatorname{curl} X$ *is* a solution of the Jacobi equation by Proposition 2.4.

So define a unit vector field e_3 along the curve $t \mapsto \eta(t, x)$ by the formula $e_3(t) = \omega\big(t, \eta(t, x)\big)/|\omega\big(t, \eta(t, x)\big)|$. Choose $e_1(t_o)$ and $e_2(t_o)$ such that $\{e_1(t_o), e_2(t_o), e_3(t_o)\}$ forms an oriented, orthonormal basis of $T_{\eta(t_o,x)}M$. Then define $e_1(t)$ and $e_2(t)$ in general to be the solutions of the equations

$$\frac{De_1}{dt} = -\left\langle e_1(t), \frac{De_3}{dt} \right\rangle e_3(t) \qquad \text{and} \qquad \frac{De_2}{dt} = -\left\langle e_2(t), \frac{De_3}{dt} \right\rangle e_3(t). \tag{4.49}$$

Then we will automatically have

$$\frac{d}{dt}\langle e_1, e_2 \rangle = \frac{d}{dt}\langle e_1, e_3 \rangle = \frac{d}{dt}\langle e_2, e_3 \rangle = 0.$$

Thus we get a natural orthonormal basis for each $T_{\eta(t,x)}M$, with one of the vectors always parallel to $\omega\big(t, \eta(t, x)\big)$.

For some number $t_2 \in (t_o, T)$ to be specified later, consider the space of smooth functions $f(t)$ vanishing outside $[t_o, t_2]$. Define vector fields $y_1(t) = f(t)e_1(t)$ and $y_2(t) = f(t)e_2(t)$. Using equations (4.49), we can compute that for $j = 1, 2$,

$$\left\langle \frac{Dy_j}{dt}, \frac{Dy_j}{dt} \right\rangle = \Big(\frac{df}{dt}\Big)^2 + f(t)^2 \Big\langle e_j(t), \frac{De_3}{dt} \Big\rangle^2.$$

Now let $\bar{i} = \frac{1}{2}\big(i_{[t_o,t_2]}(y_1,y_1) + i_{[t_o,t_2]}(y_2,y_2)\big)$. We find

$$\begin{aligned}\bar{i} = \int_{t_o}^{t_2} \left(\frac{df}{dt}\right)^2 + \frac{1}{2}f^2\left(\Big\langle e_1, \frac{De_3}{dt}\Big\rangle^2 + \Big\langle e_2, \frac{De_3}{dt}\Big\rangle^2\right) \\ - \frac{1}{2}f^2\Big(\langle \nabla_{e_1}\nabla p + R(e_1,\dot\eta)\dot\eta, e_1\rangle + \langle \nabla_{e_2}\nabla p + R(e_2,\dot\eta)\dot\eta, e_2\rangle\Big)\,dt.\end{aligned} \tag{4.50}$$

The second term in (4.50) is simply

$$\Big\langle e_1, \frac{De_3}{dt}\Big\rangle^2 + \Big\langle e_2, \frac{De_3}{dt}\Big\rangle^2 = \Big\langle \frac{De_3}{dt}, \frac{De_3}{dt}\Big\rangle. \tag{4.51}$$

The term on the second line of (4.50) can be rewritten using

$$\begin{aligned}&\langle \nabla_{e_1}\nabla p + R(e_1,\dot\eta)\dot\eta, e_1\rangle + \langle \nabla_{e_2}\nabla p + R(e_2,\dot\eta)\dot\eta, e_2\rangle - \big(\Delta p + \mathrm{Ric}(X)\big)\circ\eta \\ &\qquad = -\langle \nabla_{e_3}\nabla p + R(e_3,\dot\eta)\dot\eta, e_3\rangle = -\frac{1}{|\omega|}\langle \nabla_\omega \nabla p + R(\omega,\dot\eta)\dot\eta, e_3\rangle = \frac{1}{|\omega|}\left\langle \frac{D^2\omega}{dt^2}, e_3\right\rangle,\end{aligned}$$

using the fact that ω is a solution of the Jacobi equation, by Proposition 2.4.

Next we compute

$$\Big\langle \frac{D^2\omega}{dt^2}, e_3\Big\rangle = \frac{d^2|\omega|}{dt^2} + 2\,\frac{d|\omega|}{dt}\Big\langle \frac{De_3}{dt}, e_3\Big\rangle + |\omega|\Big\langle \frac{D^2e_3}{dt^2}, e_3\Big\rangle = \frac{d^2|\omega|}{dt^2} - |\omega|\Big\langle \frac{De_3}{dt}, \frac{De_3}{dt}\Big\rangle.$$

Inserting this expression into the expression above, we obtain the formula

$$\begin{aligned}&\langle \nabla_{e_1}\nabla p + R(e_1,\dot\eta)\dot\eta, e_1\rangle + \langle \nabla_{e_2}\nabla p + R(e_2,\dot\eta)\dot\eta, e_2\rangle \\ &\qquad = \big(\Delta p + \mathrm{Ric}(X)\big)\circ\eta + \frac{1}{|\omega|}\,\frac{d^2|\omega|}{dt^2} - \Big\langle \frac{De_3}{dt}, \frac{De_3}{dt}\Big\rangle.\end{aligned} \tag{4.52}$$

We then observe that since

$$\frac{D\omega}{dt} = \nabla_\omega X = \frac{1}{2}\,\mathrm{Def}\,X(\omega) + \frac{1}{2}\omega\times\omega = \frac{1}{2}\,\mathrm{Def}\,X(\omega),$$

and

$$\frac{1}{|\omega|}\,\frac{d|\omega|}{dt} = \frac{1}{|\omega|^2}\Big\langle \frac{D\omega}{dt}, \omega\Big\rangle = \frac{1}{2}\langle \mathrm{Def}\,X(e_3), e_3\rangle,$$

we can compute

$$\begin{aligned}\Big|\frac{De_3}{dt}\Big|^2 &= \Big|\frac{1}{|\omega|}\,\frac{D\omega}{dt} - \frac{1}{|\omega|}\,\frac{d|\omega|}{dt}e_3\Big|^2 = \frac{1}{4}\big|\,\mathrm{Def}\,X(e_3) - \langle \mathrm{Def}\,X(e_3), e_3\rangle e_3\big|^2 \\ &= \frac{1}{4}\Big(\langle \mathrm{Def}\,X(e_3), \mathrm{Def}\,X(e_3)\rangle - \langle \mathrm{Def}\,X(e_3), e_3\rangle^2\Big) = \frac{1}{4}\big|e_3\times \mathrm{Def}\,X(e_3)\big|^2.\end{aligned} \tag{4.53}$$

Finally, substituting (4.51), (4.52), and (4.53) into (4.50), we obtain

$$\bar{i} = \int_{t_o}^{t_2} \left(\frac{df}{dt}\right)^2 + f^2\Big(\frac{1}{4}\big|e_3\times \mathrm{Def}\,X(e_3)\big|^2 - \frac{1}{2}\big(\Delta p + \mathrm{Ric}(X)\big)\circ\eta - \frac{1}{2|\omega|}\,\frac{d^2|\omega|}{dt^2}\Big)\,dt. \tag{4.54}$$

To finish the proof, we now define a new function g by $g = |\omega|^{1/2} f$. Then

$$\begin{aligned}\int_{t_o}^{t_2} \Big(\frac{df}{dt}\Big)^2\,dt &= \int_{t_o}^{t_2} \frac{1}{|\omega|}\Big(\frac{dg}{dt}\Big)^2 - \frac{1}{|\omega|^2}\frac{d|\omega|}{dt} g \frac{dg}{dt} + \frac{1}{4|\omega|^3}\Big(\frac{d|\omega|}{dt}\Big)^2 g^2\,dt \\ &= \int_{t_o}^{t_2} \frac{1}{|\omega|}\Big(\frac{dg}{dt}\Big)^2 + \Big(\frac{1}{2|\omega|^2}\frac{d^2|\omega|}{dt^2} - \frac{3}{4|\omega|^3}\Big(\frac{d|\omega|}{dt}\Big)^2\Big) g^2\,dt\end{aligned} \tag{4.55}$$

after integrating the middle term by parts.

Inserting (4.55) into (4.54), we simplify $\bar{i}$ to

(4.56)

$$\begin{aligned}\bar{i} &= \int_{t_o}^{t_2} \frac{1}{|\omega|}\Big(\frac{dg}{dt}\Big)^2 + \Big(\frac{1}{4|\omega|}\big|e_3 \times \operatorname{Def} X(e_3)\big|^2 - \frac{3}{4|\omega|^3}\Big(\frac{d|\omega|}{dt}\Big)^2 - \frac{\Delta p + \operatorname{Ric}(X)}{2|\omega|}\Big) g^2\,dt \\ &= \int_{t_o}^{t_2} \frac{1}{|\omega(t)|}\Big(\frac{dg}{dt}\Big)^2 - |\omega(t)|\Psi(t) g(t)^2\,dt.\end{aligned}$$

Finally, changing variables to $s = h(t)$ yields

$$\bar{i} = \int_{h(t_o)}^{h(t_2)} \Big(\frac{dg}{ds}\Big)^2 - \Psi\big(h^{-1}(s)\big) g(s)^2\,ds. \tag{4.57}$$

This expression can be made negative for some g and some t_2 if and only if the equation (4.47) is oscillatory, by standard index theory for second-order equations.

Therefore, if equation (4.47) is oscillatory, we can make $\bar{i}$ negative for some choice of f, and therefore at least one of $i_{[t_o,t_2]}(y_1, y_1)$ or $i_{[t_o,t_2]}(y_2, y_2)$ must be negative. This completes the proof. □

Very little is known about fluids near a critical time. Aside from conditions similar to the Beale-Kato-Majda condition (4.45), the only rigorous information about blowup of the Euler equation is the condition of Constantin, Fefferman, and Majda [CFM], which is a constraint on the spatial variation of the vorticity in a neighborhood of x rather than on the various quantities *at* x. It is not clear whether the Constantin-Fefferman-Majda condition tells us anything about the function $\Psi(t)$ in (4.46).

Nonetheless we can describe some simple conditions on $\Psi(t)$ which will ensure that equation (4.47) is oscillatory. There are a wide variety of conditions we might consider; many are described in Swanson [Sw]. But the simplest one by far is the following:

Proposition 4.4. *If* $\liminf_{t\to T} \Psi(t) > 0$*, then equation* (4.47) *is oscillatory.*

Now consider the following possible geometric features of a solution X that satisfies the Beale-Kato-Majda criterion (4.45) at a point $x \in M$.

Conjecture 4.5 (Strong Conjugate Conjecture)**.** We conjecture that if the Beale-Kato-Majda criterion (4.45) is satisfied for some x in the interior of M^3 and some time T, then in addition the following conditions will also be satisfied (in what follows, all quantities are evaluated at time t and position $\eta(t, x)$).

(I) $\liminf_{t\to T} \dfrac{\Delta p + \operatorname{Ric}(X)}{|\omega|^2} \ge 0.$

(II) $\lim_{t\to T} \dfrac{|\omega \times \operatorname{Def} X(\omega)|}{|\omega|^3} = 0.$

(III) $\liminf_{t\to T} \dfrac{1}{|\omega|^2}\dfrac{d|\omega|}{dt} > 0.$

Clearly the Strong Conjugate Conjecture implies the Weak Conjugate Conjecture. Numerical computations of solutions of the Euler equations which appear to exhibit the Beale-Kato-Majda blowup condition also appear to satisfy the Strong Conjugate Conjecture.

Condition (I) is actually a local condition, by formula (2.16) for the pressure in terms of the deformation tensor and curl. So it is equivalent to saying that the vorticity grows faster than the strain rates. Condition (I) is generally believed to hold, according to Ohkitani and Kishiba [OK]. It is also believed that the term $\Delta p + \operatorname{Ric}(X)$ is much smaller in magnitude than $|\omega|^2$.

Condition (II) essentially states that the vorticity aligns with an eigenvector of the strain tensor. If condition (I) holds, then the size of the stress tensor satisfies

$$\|\operatorname{Def} X\| = \sqrt{\operatorname{Tr}(\operatorname{Def} X)^2} \le \sqrt{2}|\omega|,$$

and thus

$$\lim_{t\to T} \frac{|\omega \times \operatorname{Def} X(\omega)|}{|\omega|^3} \le 2 \lim_{t\to T} \frac{|e_3 \times \operatorname{Def} X(e_3)|}{\|\operatorname{Def} X\|},$$

so that this is basically a question of whether the normalized matrix $\operatorname{Def} X/\|\operatorname{Def} X\|$ has certain off-diagonal components approaching zero. Condition (II) is also believed to hold, according to numerical experiments of Pumir and Siggia [PS].

Condition (III) is related to the Beale-Kato-Majda criterion, but is not implied by it. For example, if $|\omega\big(t, \eta(t,x)\big)| \propto \frac{1}{1-t/T} \ln\Big(\frac{1}{1-t/T}\Big)$, then (4.45) is satisfied but condition (III) is not. On the other hand it is generally believed that the dependence of $|\omega\big(t,\eta(t,x)\big)|$ on t is algebraic, so that $|\omega(t,x)| \propto (1-t/T)^{-\gamma}$ for some number γ. We must have $\gamma \ge 1$ for (4.45) to hold. If $\gamma = 1$, then the limit in condition (III) is some positive number, while if $\gamma > 1$, the limit is positive infinity. It is widely believed that $\gamma = 1$, according to numerical experiments of Bell and Marcus [BM] and Kerr [K].

If conditions (I)–(III) hold, then we can obtain a stronger nonexistence result.

Theorem 4.6. *Suppose η is a geodesic in $\mathcal{D}_\mu(M^3)$ which exists on a time interval $[0,T)$. Suppose that for some $x \in$ int(M), the Beale-Kato-Majda condition* (4.45) *holds. Suppose also that for this T and x, the Weak Conjugate Conjecture is valid.*

Then η cannot be extended to any larger interval containing T in any class of generalized solutions which minimizes the energy integrals

$$\int_a^b \int_M \Big\langle \frac{\partial \eta}{\partial t}, \frac{\partial \eta}{\partial t} \Big\rangle \, d\mu \, dt$$

locally in time.

Proof. If the Weak Conjugate Conjecture holds, then Theorem 4.3 holds, so that for any $t_o < T$, there is a $t_1 \in (t_o, T)$ such that η is not minimizing on (t_o, t_1). Thus any extension of η which agrees with η on $[0, T)$ is not minimizing in any open interval containing T. □

The original theorem of Beale-Kato-Majda [BKM] ensured that if (4.45) held, there were no solutions which were global in time and H^3 in space. However, it is believed that there may be weak solutions that are global in time.

For example, Brenier [B1] defined a class of generalized flows which contains as special cases the paths in $\mathcal{D}_\mu(M)$, proving that the problem of the shortest path joining two diffeomorphisms always has a solution. Shnirelman used these flows to answer a number of questions about the global topology of $\mathcal{D}_\mu(M)$, showing in particular that such generalized flows could be approximated by paths in $\mathcal{D}_\mu(M)$. Thus the question of whether a generalized flow minimizes energy can be studied using the conjugate point technique, as above.

Later, Brenier [B2] related such generalized flows to the weak solutions of the Euler equation constructed by DiPerna and Majda [DM]. Since the DiPerna-Majda class is believed to be the appropriate one in which to construct weak global solutions of the Euler equations, it is reasonable to expect that any weak solution of the Euler equations should minimize energy, locally in time.

Therefore, if one could prove the stronger form of the Beale-Kato-Majda criterion (4.45) and either the Weak Conjugate Conjecture 4.2 or the Strong Conjugate Conjecture 4.5, and in addition one could show global existence in a class of solutions satisfying local minimization of energy, then Theorem 4.6 would imply global C^∞ existence of solutions to the 3-D Euler equations.

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