Solution to "A least squares plane," pg. 988, #42

Question The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k):

$$(0,0,0), \qquad (0,1,1), \qquad (1,1,1), \qquad (1,0,-1).$$

Find the values of A, B, and C that minimize the sum

$$\sum_{k=1}^{4}(Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

Answer We are minimizing the error function

$$\begin{aligned} f(A,B,C) = (Ax_1 + By_1 + C - z_1)^2 &+ (Ax_2 + By_2 + C - z_2)^2 \\ &+ (Ax_3 + By_3 + C - z_3)^2 + (Ax_4 + By_4 + C - z_4)^2 \end{aligned}$$

Plugging in $(x_1, y_1, z_1) = (0,0,0)$, etc., we get

$$f(A,B,C) = C^2 + (B+C-1)^2 + (A+B+C-1)^2 + (A+C+1)^2$$

To minimize this with respect to the three variables A, B, and C, we first find the critical points. So we solve the three equations

$$\frac{\partial f}{\partial A} = 0, \qquad \frac{\partial f}{\partial B} = 0, \qquad \frac{\partial f}{\partial C} = 0$$

These derivatives are easy to compute:

$$\begin{aligned} \frac{\partial f}{\partial A} &= 2(A+B+C-1) + 2(A+C+1) = 0 \\ \frac{\partial f}{\partial B} &= 2(B+C-1) + 2(A+B+C-1) = 0 \\ \frac{\partial f}{\partial C} &= 2C + 2(B+C-1) + 2(A+B+C-1) + 2(A+C+1) = 0 \end{aligned}$$

All these equations have an extraneous factor of 2, which we can just cancel. Simplifying what remains, we have the three equations

$$2A + B + 2C = 0 \tag{1}$$
$$A + 2B + 2C = 2 \tag{2}$$
$$2A + 2B + 4C = 1 \tag{3}$$

Let's eliminate B using equation (1). We get $B = -2A - 2C$. Plug this in to equations (2) and (3). Then we have

$$A - 4A - 4C + 2C = 2 \tag{4}$$
$$2A - 4A - 4C + 4C = 1 \tag{5}$$

Equation (5) gives $A = -\frac{1}{2}$. Then equation (4) becomes

$$-3A - 2C = 2$$

which yields $C = -\frac{1}{4}$. Finally (1) gives us $B = \frac{3}{2}$.

Because the function f is a sum of squares, its range must consist only of positive numbers. So there must be a minimum positive value of f, and that must occur at this critical point $(A, B, C) = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$.