Solutions to Second Exam, Math 114, Fall 2002

Question 1 Which of the following points lies on the same plane as $(1, 2, 0)$, $(2, 2, 1)$, and $(0, 1, 1)$?

(A) $(1, 1, 1)$ (B) $(4, -1, 1)$ (C) $(4, 1, 1)$

(D) $(1, 1, -1)$ (E) $(2, -1, 3)$ (F) $(2, 1, 3)$

Answer 1 We use $P_0 = (1, 2, 0)$ as the base point. Then vectors parallel to the plane are given by

$$\begin{aligned}\vec{u} &= (2-1)\widehat{\mathbf{\imath}} + (2-2)\widehat{\mathbf{\jmath}} + (1-0)\widehat{\mathbf{k}} \\ &= \widehat{\mathbf{\imath}} + \widehat{\mathbf{k}} \\ \vec{v} &= (0-1)\widehat{\mathbf{\imath}} + (1-2)\widehat{\mathbf{\jmath}} + (1-0)\widehat{\mathbf{k}} \\ &= -\widehat{\mathbf{\imath}} - \widehat{\mathbf{\jmath}} + \widehat{\mathbf{k}}.\end{aligned}$$

The normal vector to the plane is given by the cross-product, $\vec{n} = \vec{u} \times \vec{v}$.

$$\vec{n} = \begin{vmatrix} \widehat{\mathbf{\imath}} & \widehat{\mathbf{\jmath}} & \widehat{\mathbf{k}} \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \widehat{\mathbf{\imath}} - 2\widehat{\mathbf{\jmath}} - \widehat{\mathbf{k}}$$

So the plane is given by the equation $\vec{n} \cdot \overrightarrow{P_0P} = 0$, which is

$$(x-1) - 2(y-2) - (z-0) = 0$$

or more simply

$$x + 3 = 2y + z.$$

By inspection we see that the only solution is the point $(2, 1, 3)$ corresponding to choice **(F)**.

Question 2 Consider the function

$$f(x,y) = x^2 - 2x\cos(xy)$$

Which of the following statements correctly describes the behavior of $f(x,y)$?

(I) f has infinitely many critical points.

(II) f has a saddle point at $(1, 2\pi)$.

(III) f has a local maximum at $(1, 0)$.

(IV) f has a local minimum at $(-1, \pi)$.

(A) only (II) (B) (I) and (II) (C) (III) and (IV)

(D) only (III) (E) (I) and (IV) (F) (I), (II) and (IV)

Answer 2 First we compute the critical points. The equation $f_x = 0$ implies

$$2x - 2\cos(xy) + 2xy\sin(xy) = 0$$

while $f_y = 0$ implies

$$2x^2\sin(xy) = 0$$

From this second equation, we have either $x = 0$ or $\sin(xy) = 0$. If $x = 0$, then the first equation implies $\cos 0 = 0$, which is impossible. So the only solutions are when $\sin(xy) = 0$. This happens when $xy = n\pi$ for some integer n. Then we have $\cos(xy) = \cos(n\pi) = (-1)^n$. So $f_x = 0$ becomes

$$2x - 2(-1)^n = 0$$

which implies $x = (-1)^n$, so $y = (-1)^n n\pi$. Thus we have infinitely many critical points, so part (I) is true.

Now we compute the second derivatives to determine information the local behavior of the critical points.

$$f_{xx} = 2 + 4y\sin(xy) + 2xy^2\cos(xy), \quad f_{xy} = 4x\sin(xy) + 2x^2y\cos(xy), \quad f_{yy} = 2x^3\cos(xy)$$

At the point $x = (-1)^n$, $y = (-1)^n n\pi$, we have

$$f_{xx} = 2 + 2n^2\pi^2, \quad f_{xy} = 2n\pi, \quad f_{yy} = 2$$

Since $f_{xx} > 0$ for any n, and since

$$f_{xx}f_{yy} - f_{xy}^2 = 4 + 4n^2\pi^2 - 4n^2\pi^2 = 4 > 0$$

for any n, every critical point is a local minimum.

So part (IV) is true and parts (II) and (III) are false. Thus the correct choice is **(E)**.

Question 3 Find the point at which the plane passing through $P = (1, 0, -1)$ and perpendicular to the line $\overrightarrow{r}(t) = (2-t)\widehat{\imath} + (2+t)\widehat{\jmath} + t\widehat{\mathbf{k}}$ intersects the line.

(A) $\left(\frac{1}{2}, \frac{7}{2}, \frac{3}{2}\right)$ (B) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{2}{3}\right)$ (C) $\left(\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right)$

(D) $\left(\frac{5}{3}, \frac{7}{3}, \frac{1}{3}\right)$ (E) $(3, 1, -1)$ (F) $(-1, 5, 3)$

Answer 3 A vector parallel to the line

$$x = 2 - t, \qquad y = 2 + t, \qquad z = t$$

is $\overrightarrow{u} = -\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}$ (just take the coefficients of t). So the normal vector to the desired plane is $\overrightarrow{n} = -\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}$.

The plane with normal vector $\overrightarrow{n} = -\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}$ and passing through $P = (1, 0, -1)$ is given by

$$(-1)(x-1) + (1)(y) + (1)(z+1) = 0$$

which simplifies to

$$-x + y + z + 2 = 0$$

When does the line intersect this plane? Plug in the equations of the line:

$$\begin{aligned} -(2-t) + (2+t) + t + 2 &= 0 \\ 3t + 2 &= 0 \\ t &= -\tfrac{2}{3} \end{aligned}$$

So $x = 2 + \frac{2}{3} = \frac{8}{3}$, $y = 2 - \frac{2}{3} = \frac{4}{3}$, and $z = -\frac{2}{3}$. So the correct choice is (B).

Question 4 Which of these functions satisfies the equation

$$f_{xx} + f_{yy} = -f$$

(I) $f(x, y) = \sin(xy)$

(II) $f(x, y) = \sin x + \sin y$

(III) $f(x, y) = \sin(x + y)$

(IV) $f(x, y) = \sin x \sin y$

(A) (I) only (B) (II) only (C) (II) and (III)

(D) (III) and (IV) (E) (I), (II), (IV) (F) (I), (III), (IV)

Answer 4 We compute f_{xx} and f_{yy} for each choice.

- (I) $f_{xx} = -y^2 \sin(xy)$ and $f_{yy} = -x^2 \sin(xy)$, so $f_{xx} + f_{yy} = -(x^2 + y^2)\sin(xy) \neq -f$. So (I) is false.
- (II) $f_{xx} = -\sin x$ and $f_{yy} = -\sin y$, so $f_{xx} + f_{yy} = -\sin x - \sin y = -f$. So (II) is true.
- (III) $f_{xx} = -\sin(x + y)$ and $f_{yy} = -\sin(x + y)$, so $f_{xx} + f_{yy} = -2\sin(x + y) \neq -f$. So (III) is false.
- (IV) $f_{xx} = -\sin x \sin y$ and $f_{yy} = -\sin x \sin y$, so $f_{xx} + f_{yy} = -2\sin x \sin y \neq -f$. So (IV) is false.

Thus the correct answer is **(B)**.

Question 5 The line of intersection of the two tangent planes to the surfaces $xyz = 1$ and $x^2 + 3y^2 + z^2 = 5$ at the point $P = (1, -1, -1)$ intersects the yz-plane at the point

(A) $(0, 1, -1)$	(B) $(0, -1, 1)$	(C) $(0, 2, -1)$
(D) $(0, -1, 2)$	(E) $(0, -1, -2)$	(F) $(0, -2, -2)$

Answer 5 To find the line of intersection, we need to find a vector $\vec{v}$ to which it is parallel. We know this vector lies in both tangent planes, so it must be perpendicular to the normals of both planes. So if $\vec{n_1}$ and $\vec{n_2}$ are the normals to the two planes, $\vec{v} = \vec{n_1} \times \vec{n_2}$

To find $\vec{n_1}$ and $\vec{n_2}$, we use the fact that the gradient is always perpendicular to the level surfaces. So

$$\vec{n_1} = \nabla(xyz)\Big|_{(1,-1,-1)} = \left(yz\widehat{\imath} + xz\widehat{\jmath} + xy\widehat{\mathbf{k}}\right)\Big|_{(1,-1,-1)} = \widehat{\imath} - \widehat{\jmath} - \widehat{\mathbf{k}}$$

and

$$\vec{n_2} = \nabla(x^2 + 3y^2 + z^2)\Big|_{(1,-1,-1)} = \left(2x\widehat{\imath} + 6y\widehat{\jmath} + 2z\widehat{\mathbf{k}}\right)\Big|_{(1,-1,-1)} = 2\widehat{\imath} - 6\widehat{\jmath} - 2\widehat{\mathbf{k}}$$

The cross product of these two vectors is

$$\vec{v} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{\mathbf{k}} \\ 1 & -1 & -1 \\ 2 & -6 & -2 \end{vmatrix} = -4\widehat{\imath} - 4\widehat{\mathbf{k}}$$

So the line parallel to $-4\widehat{\imath} - 4\widehat{\mathbf{k}}$ and passing through $(1, -1, -1)$ is given parametrically by the equations

$$x = 1 - 4t, \qquad y = -1, \qquad z = -1 - 4t$$

This crosses the plane $x = 0$ when $t = \frac{1}{4}$.

At $t = \frac{1}{4}$, we have $x = 0$, $y = -1$, and $z = -2$. So the answer is **(E)**.

Question 6 Find the length of the shortest segment which joins a point on the curve

$$x = -s^2, \qquad y = s$$

to a point on the curve

$$x = 2t, \qquad y = 3 - t.$$

Hint: Minimize the square of the distance as a function of s and t.

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2

(D) $\sqrt{5}$ (E) $\sqrt{6}$ (F) $\sqrt{7}$

Answer 6 The square of the distance is

$$\begin{aligned} f(s,t) &= [x(t) - x(s)]^2 + [y(t) - y(s)]^2 \\ &= (2t + s^2)^2 + (3 - t - s)^2 \end{aligned}$$

To minimize f, we find the critical points by setting the partial derivatives f_s and f_t equal to zero.

$$\begin{aligned} \frac{\partial f}{\partial s} &= 2(2t + s^2)(2s) + 2(3 - t - s)(-1) = 0 \\ \frac{\partial f}{\partial t} &= 2(2t + s^2)(2) + 2(3 - t - s)(-1) = 0 \end{aligned}$$

If we subtract the second equation from the first, we obtain

$$2(2t + s^2)(2s - 2) = 0$$

which implies that either $2t + s^2 = 0$ or that $s = 1$. $2t + s^2 = 0$ implies that $t + s = 3$, so that $s^2 - 2s + 6 = 0$. This equation has no solution for s, so $2t + s^2 = 0$ is impossible. So $s = 1$.

Plugging $s = 1$ into the equation $f_s = 0$, we get

$$2(2t + 1) + (t - 2) = 0$$

which implies $t = 0$. So the square of the distance is minimized at $t = 0$, $s = 1$. The distance at $t = 0$, $s = 1$ is

$$\sqrt{f(1,0)} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

So the correct answer is **(D)**.

Question 7 Consider the surface given in spherical coordinates by

$$\rho = \frac{2\cos\phi}{\cos(2\phi)}, \qquad 0 \leq \phi \leq \frac{\pi}{4}$$

Which of the following is the cartesian equation of the curve at which this surface intersects the plane $z = 3$? Hint: $\cos 2\phi = \cos^2\phi - \sin^2\phi$.

(A) $x^2 + y^2 = 0$ (B) $x^2 - y = 2$ (C) $x^2 - y^2 = 3$

(D) $x^2 + y^2 = 3$ (E) $x^2 - y^2 = 0$ (F) $(x + y)^2 = 1$

Answer 7 First we obviously use the hint, then we look for terms that can be eliminated using the formulas

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

We obtain

$$\begin{aligned} \rho &= \frac{2\cos\phi}{\cos^2\phi - \sin^2\phi} \\ \rho(\cos^2\phi - \sin^2\phi) &= 2\cos\phi \\ \rho^2\cos^2\phi - \rho^2\sin^2\phi &= 2\rho\cos\phi \\ z^2 - (x^2 + y^2) &= 2z \end{aligned}$$

Now using the equation $z = 3$, we get

$$x^2 + y^2 = z^2 - 2z = 3$$

So the correct answer is choice **(D)**.

Question 8 The absolute maximum of the function $f(x, y) = xy$ subject to the constraint

$$9x^2 + 4y^2 \leq 36$$

is

(A) 3　　(B) 15　　(C) $2\sqrt{2}$

(D) 9　　(E) $12\sqrt{2}$　　(F) does not exist

Answer 8 To determine the absolute maximum, we first determine the critical points of f inside the ellipse. So we solve the equations $f_x = y = 0$ and $f_y = x = 0$ to get $(0, 0)$ as one critical point.

Then we maximize $f(x, y) = xy$ on the curve $g(x, y) = 9x^2 + 4y^2 = 36$. Using the Lagrange multiplier method, we get

$$\begin{aligned} y &= 18\lambda x \\ x &= 8\lambda y \end{aligned}$$

The solution is $x = 0$, $y = 0$, which does not satisfy the constraint, or $\lambda = \pm\frac{1}{12}$.

If $\lambda = \frac{1}{12}$, then $y = \frac{3}{2}x$, and plugging this into the constraint gives $9x^2 + 9x^2 = 36$, so $x = \pm\sqrt{2}$. So the critical points here are $(\sqrt{2}, \frac{3}{2}\sqrt{2})$ and $(-\sqrt{2}, -\frac{3}{2}\sqrt{2})$.

If $\lambda = -\frac{1}{12}$, then $y = -\frac{3}{2}x$, and the same computation yields the critical points $(\sqrt{2}, -\frac{3}{2}\sqrt{2})$ and $(-\sqrt{2}, \frac{3}{2}\sqrt{2})$.

Now we plug the five critical points into f. We get

$$f(0, 0) = 0, \qquad f(\pm\sqrt{2}, \pm\tfrac{3}{2}\sqrt{2}) = 3, \qquad f(\pm\sqrt{2}, \mp\tfrac{3}{2}\sqrt{2}) = -3$$

So the absolute maximum is 3, and thus the correct choice is **(A)**.

Question 9 Find the directional derivative of

$$f(x,y,z)=\cos(xy)+e^{yz}-\ln(xz)$$

at the point $(1,0,2)$ in the direction from $(1,0,2)$ to $(2,3,4)$.

Answer 9 First we find a unit vector in the desired direction. The vector is $\vec{v}=\widehat{\mathbf{\imath}}+3\widehat{\mathbf{\jmath}}+2\widehat{\mathbf{k}}$, and the length is $|\vec{v}|=\sqrt{1^2+3^2+2^2}=\sqrt{14}$. So the desired unit vector is

$$\vec{u}=\frac{1}{\sqrt{14}}(\widehat{\mathbf{\imath}}+3\widehat{\mathbf{\jmath}}+2\widehat{\mathbf{k}})$$

Now we need the gradient of the function:

$$\begin{aligned}\nabla f&=\frac{\partial f}{\partial x}\widehat{\mathbf{\imath}}+\frac{\partial f}{\partial y}\widehat{\mathbf{\jmath}}+\frac{\partial f}{\partial z}\widehat{\mathbf{k}}\\&=\left[-y\sin(xy)-\tfrac{1}{x}\right]\widehat{\mathbf{\imath}}+\left[-x\sin(xy)+ze^{yz}\right]\widehat{\mathbf{\jmath}}+\left[ye^{yz}-\tfrac{1}{z}\right]\widehat{\mathbf{k}}\end{aligned}$$

So at the point $(1,0,2)$, we have

$$\nabla f\Big|_{(1,0,2)}=-\widehat{\mathbf{\imath}}+2\widehat{\mathbf{\jmath}}-2\widehat{\mathbf{k}}$$

Thus the directional derivative is

$$D_{\vec{u}}f=\nabla f\cdot\vec{u}=\boldsymbol{\frac{1}{\sqrt{14}}}$$

Question 10 Suppose $f(x, y)$ is a smooth function. Suppose $f(1, 2) \approx 3.1$, $f(1.1, 2) \approx 3.4$, and $f(1, 2.2) \approx 2.9$. Find the best estimate for $f(0.9, 2.1)$ using linear approximation.

Answer 10 We want to find the linear approximation at $(x_0, y_0) = (1, 2)$ of f, so we use

$$L(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)$$

We need to estimate f_x and f_y. To do this, write

$$f_x(1, 2) \approx \frac{f(1.1, 2) - f(1, 2)}{0.1} = 3$$

and

$$f_y(1, 2) \approx \frac{f(1, 2.2) - f(1, 2)}{0.2} = -1$$

Then we have

$$L(x, y) \approx 3.1 + 3(x - 1) - (y - 2)$$

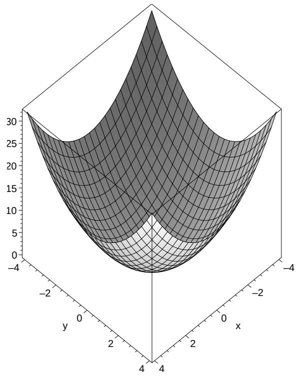
So our approximation is

$$f(0.9, 2.1) \approx 3.1 + 3(-0.1) - (0.1) = \mathbf{2.7}$$

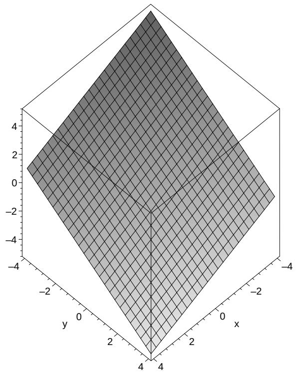
Question 11 Match the equations to their graphs. Enter your answers in the table at the bottom of the page and show your reasoning on the next page.

(1) $z = \sin(x)y$ (2) $z = y^3$ (3) $z = x^2 + y^2$

(4) $2x + 3y + 4z = 0$ (5) $z = \cos\left(\frac{\pi}{1+x^2+y^2}\right)$ (6) $z = x^2$



(a)



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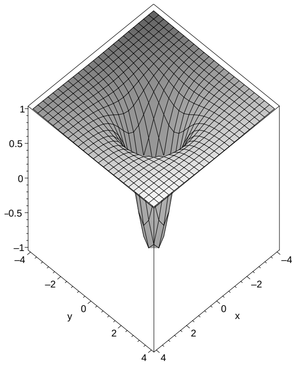

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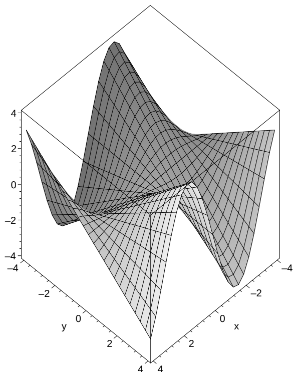

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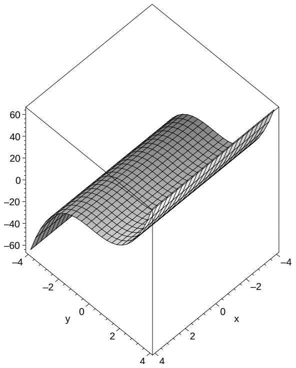

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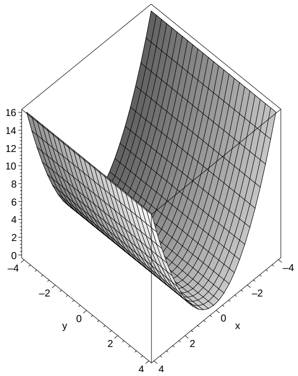

(f)

Answer 11

(a) There are only two rotationally symmetric graphs, this one and (c). Since this one clearly resembles a paraboloid of revolution, it must be $z = x^2 + y^2$, choice (3).

(b) This is the only planar graph, so it must be $2x + 3y + 4z = 0$, choice (4).

(c) This is the other rotationally symmetric graph, so it must be choice (5).

(d) The graph is not obvious, until we look at the intersection with surfaces of constant y and see the sine graph. So it must be $z = \sin(x)y$, choice (1).

(e) The profile is the same for all x, so the function must depend only on y, and must be $z = y^3$, choice (2).

(f) The profile is the same for all y, so it depends only on x. So it is $z = x^2$, choice (6).

Question 12 True or false. Explain your reasoning.

(a) The line $\vec{r}(t) = (1+2t)\hat{\imath}+(1+3t)\hat{\jmath}+(1+4t)\widehat{\mathbf{k}}$ is perpendicular to the plane $2x+3y-4z = 9$.

(b) The equation $r = 3z$ in cylindrical coordinates, describes a cone.

(c) $|\vec{a} \times \vec{b}| = 0$ implies that either $\vec{a} = 0$ or $\vec{b} = 0$.

Answer 12

(a) This is **false**. The line is perpendicular to the plane if and only if the vector parallel to the line is perpendicular to the plane. This happens if and only if the vector along the line is a constant multiple of the normal vector of the plane.

The vector along the line is $\vec{v} = 2\hat{\imath} + 3\hat{\jmath} + 4\widehat{\mathbf{k}}$. The vector normal to the plane is $\vec{n} = 2\hat{\imath}+3\hat{\jmath}-4\widehat{\mathbf{k}}$. Since $\vec{v}$ is not parallel to $\vec{n}$, this line is not perpendicular to the plane.

(b) This is **true**. In rectangular coordinates,

$$r = 3z \implies r^2 = 9z^2 \implies x^2 + y^2 = 9z^2$$

which is the equation of a cone through the origin, compressed by a factor of 3 in the z direction.

(c) This is **false**. $|\vec{a} \times \vec{b}| = 0$ may be zero if $\vec{a}$ is parallel to $\vec{b}$. For example, if $\vec{a} = \hat{\imath}$ and $\vec{b} = \hat{\imath}$, then $\vec{a} \times \vec{b} = 0$.