Solutions to Final Exam, Math 114, Fall 2002

Question 1 Find the point at which the tangent line to the curve

$$x = 3t^2 - t \qquad y = 2t + t^3$$

at $t = 1$ intersects the line $y = 2 - x$.

(A) $\left(\frac{4}{5}, \frac{6}{5}\right)$ (B) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (C) does not exist

(D) $(1, 1)$ (E) $\left(\frac{2}{3}, \frac{4}{3}\right)$ (F) $\left(\frac{3}{4}, \frac{5}{4}\right)$

Answer 1 At $t = 1$, the point on the curve is $(x_0, y_0) = (2, 3)$. The slope of the tangent line is

$$m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Since $x'(t) = 6t - 1$ and $y'(t) = 2 + 3t^2$, we have $x'(1) = 5$ and $y'(1) = 5$, so that $m = \frac{5}{5} = 1$.

Thus the tangent line is $y - 3 = 1(x - 2)$ or $y = x + 1$. The intersection between this line and the line $y = 2 - x$ is at the point $\left(\frac{1}{2}, \frac{3}{2}\right)$. So the correct answer is **(B)**.

Question 2 Find the length of the curve $r = \cos(\theta) - \sin(\theta)$, $0 \leq \theta \leq \frac{\pi}{4}$.

(A) $\frac{\pi}{2}$ (B) 1 (C) $\frac{1}{\sqrt{2}}$

(D) $\frac{\pi}{2\sqrt{2}}$ (E) $\frac{1}{4}$ (F) 2

Answer 2 The length of a curve given in polar coordinates is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

In this case we have $\frac{dr}{d\theta} = -\sin\theta - \cos\theta$ so that

$$r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 2$$

So the integral is

$$L = \int_0^{\pi/4} \sqrt{2}\, d\theta = \frac{\pi\sqrt{2}}{4}$$

Thus the correct answer is **(D)**.

Question 3 Which of the sets described by the cylindrical coordinate inequalities are unbounded? (Assume r is always positive.)

(I) $r < 1$, $z < 1$

(II) $r + z^2 < 1$

(III) $z + r^2 < 1$

(A) (I) only (B) (II) only (C) (III) only

(D) (I) and (II) (E) (I) and (III) (F) (II) and (III)

Answer 3 We work case by case.

The first set describes the bottom portion of a cylinder. r is bounded by 1, but z is free to go out to $-\infty$. So set (I) is unbounded.

For the second set, the inequality $r + z^2 < 1$ implies that $z^2 < 1$, so that $|z| < 1$. Also it implies that $r < 1$, so that r is bounded as well. Thus set (II) is bounded.

The third set is unbounded, since again z is free to go out to $-\infty$ if $r = 0$.

Thus the unbounded sets are (I) and (III), and so the correct answer is **(E)**.

Question 4 Evaluate the double integral

$$\int_0^1 \int_x^1 \sqrt{2+y^2}\, dy\, dx.$$

(A) $\frac{2}{3}$ (B) $\sqrt{3} - \frac{2}{3}\sqrt{2}$ (C) $\frac{4}{3} + \sqrt{2}$

(D) 0 (E) $\frac{1}{3}$ (F) none of the above

Answer 4 The integral is possible as written, though it is rather difficult (requiring a trigonometric substitution). It is much easier to change the order of integration first.

The region described is bounded above by $y = 1$ and below by $y = x$, between the limits $x = 0$ and $x = 1$. We can also think of this triangular region as being bounded on the left by $x = 0$ and on the right by $x = y$, between the limits $y = 0$ and $y = 1$. Then the integral becomes

$$\begin{aligned}\int_0^1 \int_x^1 \sqrt{2+y^2}\, dy\, dx &= \int_0^1 \int_0^y \sqrt{2+y^2}\, dx\, dy = \int_0^1 x\Big|_0^y \sqrt{2+y^2}\, dy \\ &= \int_0^1 y\sqrt{2+y^2}\, dy = \frac{1}{3}(2+y^2)^{3/2}\Big|_0^1 = \frac{1}{3}(3\sqrt{3} - 2\sqrt{2}) = \boldsymbol{\sqrt{3} - \frac{2}{3}\sqrt{2}}\end{aligned}$$

So the correct answer is **(B)**.

Question 5 Which of these functions could be the general solution of a linear homogeneous second-order differential equation with constant coefficients?

(I) $y(x) = C_1 e^{C_2 x}$

(II) $y(x) = C_1 \sin x + C_2$

(III) $y(x) = C_1 x e^x + C_2 x e^{2x}$

(A) (I) only	(B) (II) only	(C) (III) only
(D) (II) and (III)	(E) all of them	(F) none of them

Answer 5 The general solution of a homogeneous second-order equation is always of the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

so (I) cannot be correct.

(II) cannot be correct since if $\sin x$ were a solution of the equation, then $\cos x$ would also be a solution of the equation, since then the roots of the characteristic equation would be $r = \pm i$.

(III) cannot be correct since xe^x would appear only if $r = 1$ were a double root of the characteristic equation, in which case e^x would appear as well.

So the correct answer is "none of them," **(F)**.

Question 6 Let z be a complex number satisfying

$$\frac{1}{z} = \frac{2+i}{1-i}.$$

What is $\mathrm{Im}(z)$?

(A) $\sqrt{2}$	(B) $\sqrt{5}$	(C) $2i$
(D) $-\frac{3}{5}$	(E) $\frac{1}{5}i$	(F) $\frac{2}{5}$

Answer 6 Taking reciprocals, we have

$$z = \frac{1-i}{2+i}$$

Now we eliminate the complex number in the denominator by conjugating:

$$z = \frac{1-i}{2+i}\,\frac{2-i}{2-i} = \frac{2-2i-i+i^2}{4-2i+2i+1} = \frac{1-3i}{5}$$

So $\mathrm{Im}(z)$, the imaginary part of z, is $-\frac{3}{5}$. Thus the correct answer is **(D)**.

Question 7 Compute

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sin\left(\pi[x^2+y^2]\right)\,dy\,dx$$

using polar coordinates.

(A) 0 (B) $\frac{\pi}{2}$ (C) π

(D) 1 (E) $\pi\sqrt{2}$ (F) 2π

Answer 7 The region is described by the inequalities $-1 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$. This represents the top half of a circle. We can thus write it in polar coordinates as $0 \le r \le 1$ and $0 \le \theta \le \pi$.

The area element $dy\,dx$ becomes $r\,dr\,d\theta$, and so the new integral is

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sin\left(\pi[x^2+y^2]\right)\,dy\,dx &= \int_0^{\pi}\int_0^1 \sin\left(\pi r^2\right) r\,dr\,d\theta\\ &= \int_0^{\pi} -\frac{1}{2\pi}\cos\left(\pi r^2\right)\Big|_0^1\,d\theta\\ &= \int_0^{\pi}\frac{1}{\pi}\,d\theta\\ &= \mathbf{1}\end{aligned}$$

So the correct answer is **(D)**.

Question 8 Identify the functions which satisfy

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

(I) $f(x,y) = \sin x\cos y + \sin y\cos x$

(II) $f(x,y) = y\ln x + x\ln y$

(III) $f(x,y) = e^{xy}$

(A) (I) only (B) (II) only (C) (III) only

(D) (I) and (II) (E) (I) and (III) (F) (II) and (III)

Answer 8 We simply compute the partial derivatives of each function. For (I), we have $f_x = \cos x\cos y - \sin y\sin x$ and $f_y = -\sin x\sin y + \cos y\cos x$, so $f_x = f_y$ in this case.

For (II), we have $f_x = y/x + \ln y$ and $f_y = \ln x + x/y$, so $f_x \ne f_y$.

For (III), we have $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so $f_x \ne f_y$.

Thus the answer is (I) only, choice **(A)**.

Question 9 Where does the normal line to the surface $x^2 + 2y + 2z = 3$ at the point $(-1, 0, 1)$ intersect the xy-plane?

(A) $(1, 1, 0)$ (B) $(1, -1, 0)$ (C) $(0, 1, 0)$

(D) $(0, -1, 0)$ (E) $(-1, 1, 0)$ (F) $(-1, -1, 0)$

Answer 9 The normal line through the point is found by first computing the gradient vector, since the gradient is normal to the surface. For the function $f(x, y, z) = x^2 + 2y + 2z$, the gradient is

$$\nabla f = 2x\,\widehat{\mathbf{\imath}} + 2\,\widehat{\mathbf{\jmath}} + 2\,\widehat{\mathbf{k}}$$

At $(-1, 0, 1)$, $\nabla f = -2\widehat{\mathbf{\imath}} + 2\widehat{\mathbf{\jmath}} + 2\widehat{\mathbf{k}}$. So the line through the point $(-1, 0, 1)$ parallel to ∇f is

$$x = -1 - 2t, \qquad y = 0 + 2t, \qquad z = 1 + 2t$$

This line intersects the xy-plane when $z = 0$, i.e. when $t = -\frac{1}{2}$. When $t = -\frac{1}{2}$, $x = 0$ and $y = -1$. So the correct answer is $\mathbf{(0, -1, 0)}$, choice **(D)**.

Question 10 If $z = e^x \cos y$ and $x = \ln 2$ with an error of 0.1, and $y = \pi$ with an error of 0.2, what is the maximum error in z?

(A) 0 (B) 0.1 (C) 0.2

(D) 0.3 (E) 0.4 (F) 0.5

Answer 10 The linearization formula gives

$$\Delta z \approx \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y$$

Since $z_x = e^x \cos y$ and $z_y = -e^x \sin y$, at $x = \ln 2$ and $y = \pi$ we have $z_x = -2$ and $z_y = 0$. So

$$|\Delta z| \approx |-2\,\Delta x| + |0\,\Delta y| = \mathbf{0.2}$$

So the correct answer is **(C)**.

Question 11 Find the volume under the plane $z = 1 - x - y$, above the plane $z = 0$, and enclosed by the region $x \geq 0$, $y \geq 0$, $x + y \leq 1$.

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$

(D) $\frac{1}{3}$ (E) $\frac{1}{2}$ (F) 1

Answer 11 The region of integration in the plane is given by $0 \leq y \leq 1 - x$ with bounds $x = 0$ and $x = 1$. The volume is the double integral:

$$V = \iint_R (1 - x - y)\, dA = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$$
$$= \int_0^1 \left((1-x)y - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{2}(1-x)^2\, dx = -\frac{1}{6}(1-x)^3\Big|_0^1 = \mathbf{\frac{1}{6}}$$

So the correct answer is **(A)**.

Question 12 Let S be the surface given in cylindrical coordinates by the equation $z = 1 + r^2$. The intersection of S with the plane $3y - \pi x = 0$ is

(A) empty (B) a hyperbola (C) a pair of lines

(D) a point (E) a parabola (F) a circle

Answer 12 Writing S in rectangular coordinates, we get $z = 1 + x^2 + y^2$, which is obviously a paraboloid opening along the z-axis, shifted up one unit. The plane $3y - \pi x = 0$ gives a vertical slice through the paraboloid (passing through the vertex), resulting in the graph of **a parabola**. So the answer is **(E)**.

Question 13 Find the absolute maximum of $f(x, y) = 9x^2y$ on the triangle bounded by the x-axis, the y-axis, and the line $x + y = 1$.

(A) 0 (B) 9 (C) 3

(D) $\frac{4}{3}$ (E) 3 (F) $\frac{9}{2}$

Answer 13 The function's critical points occur where $f_x = 18xy = 0$ and $f_y = 9x^2 = 0$ simultaneously; that is, whenever $x = 0$. So f has critical points along the entire y-axis, but here $f(0, y) = 0$. Also on the x-axis, $f(x, 0) = 0$.

Since there are no critical points in the interior of the triangle, the maximum must occur on the boundary. And since it does not occur on either of the axes—f is positive in the first quadrant but vanishes on the axes—it must occur on the line $x + y = 1$.

We could use a Lagrange multiplier, but it's easier just to substitute $y = 1 - x$ into the equation for f, to obtain

$$f(x, 1 - x) = 9x^2(1 - x)$$

We maximize this by taking the derivative with respect to x and setting it equal to zero.

$$\frac{d}{dx} f(x, 1 - x) = 18x - 27x^2 = 0$$

implies that $x = 0$ or that $x = \frac{2}{3}$. We've already eliminated $x = 0$, so $x = \frac{2}{3}$ is the only remaining possibility.

Since $f\left(\frac{2}{3}, \frac{1}{3}\right) = \mathbf{\frac{4}{3}}$, this must be the maximum and **(D)** is the answer.

Question 14 A function $y(x)$ satisfies the initial value problem

$$xy' + y = 2x^2, \qquad y(2) = \frac{1}{6}$$

Find $y(1)$.

(A) $\frac{1}{6}$ (B) $-\frac{13}{3}$ (C) 0

(D) 1 (E) -5 (F) 11

Answer 14 The easiest way to solve this is to notice that the left-hand side is already written as the derivative of a product, so we don't even need to find an integrating factor. We just write

$$\begin{aligned}\frac{d}{dx}\big(xy(x)\big) &= 2x^2\\ xy(x) &= \int 2x^2\,dx\\ xy(x) &= \frac{2}{3}x^3 + C\\ y(x) &= \frac{2}{3}x^2 + \frac{C}{x}\end{aligned}$$

Now we find C using the initial condition $y(2) = \frac{1}{6}$, and we obtain $\frac{C}{2} = \frac{1}{6} - \frac{16}{6} = -\frac{5}{2}$ so $C = -5$. Thus

$$y(x) = \frac{2}{3}x^2 - \frac{5}{x}$$

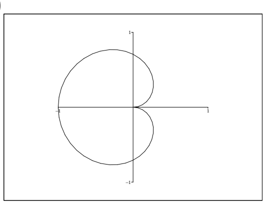
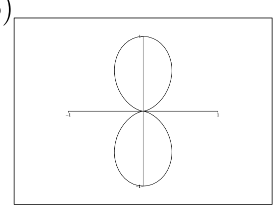
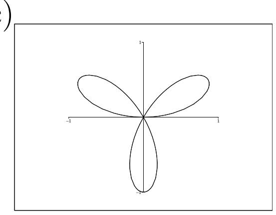
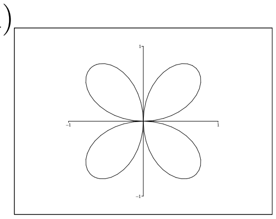
So at $x = 1$, $y(1) = \frac{2}{3} - 5 = -\frac{13}{3}$. So the correct choice is **(B)**.

Question 15 Match the polar coordinate equations to their graphs, for $0 \le \theta \le 2\pi$, $r \ge 0$. (**Hint:**it is easier to use the polar equations directly than to convert to Cartesian equations.) Enter your answers in the table below. Give a reason for each choice.

Equations:

(i) $r = \sin^2(\theta/2)$ (ii) $r = \sin(3\theta)$ (iii) $r = \sin^2(\theta)$ (iv) $r = \sin^2(2\theta)$

Graphs:

(a)

(b)

(c)

(d)


Answers:

Graphs	(a)	(b)	(c)	(d)
Equations	(i)	(iii)	(ii)	(iv)

Answer 15 We can distinguish the graphs just by plugging in the cardinal values of θ: 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Equation (i) gives $r = 0$ at $\theta = 0$, $r = \frac{1}{2}$ at $\theta = \frac{\pi}{2}$, $r = 1$ at $\theta = \pi$, and $r = \frac{1}{2}$ at $\theta = \frac{3\pi}{2}$. So it must correspond to the graph (a).

Equation (ii) gives $r = 0$ at $\theta = 0$, $r = -1$ at $\theta = \frac{\pi}{2}$ (which is undefined in the problem since we assume $r \geq 0$), $r = 0$ at $\theta = \pi$, and $r = 1$ at $\theta = \frac{3\pi}{2}$. So it must be the three-petaled flower in graph (c).

Equation (iii) gives $r = 0$ at $\theta = 0$, $r = 1$ at $\theta = \frac{\pi}{2}$, $r = 0$ at $\theta = \pi$, and $r = 1$ at $\theta = \frac{3\pi}{2}$. So it must be the two ovals in graph (a).

Equation (iv) gives $r = 0$ at any of the cardinal values of θ, so it must correspond to the four-petaled flower in graph (d).

Question 16 True or false. Explain your reasoning.

(a) If $|\vec{a} \times \vec{b}| = 3$, then $|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = 6$.

(b) If $\widehat{\mathbf{\imath}} \times \vec{a} = \vec{0}$ and $\widehat{\mathbf{\imath}} \cdot \vec{a} = -1$, then $|\vec{a}| = 1$.

Answer 16

(a) Since $\vec{a} \times \vec{a} = 0$ and $\vec{b} \times \vec{b} = 0$, the cross product can be simplified by distributing:

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = -\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = -2\vec{a} \times \vec{b}$$

Therefore

$$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = |-2\vec{a} \times \vec{b}| = 2|\vec{a} \times \vec{b}| = 6$$

Thus the statement is **true**.

(b) We know that $\vec{a}$ is parallel to $\widehat{\mathbf{\imath}}$ since their cross product is zero. Thus $\vec{a} = r\widehat{\mathbf{\imath}}$ for some scalar r. Since $\vec{a} \cdot \widehat{\mathbf{\imath}} = -1$, we must have $r = -1$. Thus

$$|\vec{a}| = |-\widehat{\mathbf{\imath}}| = 1.$$

So this statement is also **true**.

Question 17 Show that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x) - \sin(y)}{x + y}$$

does not exist.

Answer 17 The easiest way to show the limit of the function does not exist is to find two separate paths approaching the origin, such that the function approaches two different values along these paths.

First let's try approaching on the y-axis, so that $x = 0$ and $y \neq 0$. Then we have

$$\lim_{y \to 0} \frac{\sin 0 - \sin y}{0 + y} = -1$$

Now if we approach along the x-axis, so that $y = 0$ and $x \neq 0$, then we have

$$\lim_{x \to 0} \frac{\sin x - \sin 0}{x + 0} = 1$$

Since the function has two different limits along the two different paths, the limit does not exist.

Question 18 Let $f(x, y, z)$ be a function which at the point $(1, 0, 1)$ increases most rapidly in the direction of the vector $\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}$, with a rate of increase 2. Find the directional derivative of f starting at the point $(1, 0, 1)$ and going in the direction parallel to the vector $\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}}$.

Answer 18 To find the directional derivative of f in direction $\vec{u}$, we use the formula

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

So the first thing to do is to figure out ∇f. Since the gradient is the direction of most rapid increase, the problem is telling us that ∇f must be parallel to $\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}$; hence

$$\nabla f = r(\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}})$$

for some scalar r. Also since the rate of increase is 2, we know that $|\nabla f| = 2$. Therefore we have

$$2 = |\nabla f| = r|\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}| = r\sqrt{1^2 + 1^2 + 1^2} = r\sqrt{3}$$

So $r = \frac{2}{\sqrt{3}}$ and hence

$$\nabla f = \frac{2}{\sqrt{3}}(\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}})$$

The other thing to do is to find the vector $\vec{u}$. Directional derivatives are always computed along unit vectors, so we need to find a unit vector in the same direction as $\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}}$. This is

$$\vec{u} = \frac{1}{|\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}}|}(\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}}) = \frac{1}{\sqrt{3}}(\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}})$$

So we have

$$\begin{aligned} D_{\vec{u}} f &= \nabla f \cdot \vec{u} \\ &= \frac{2}{\sqrt{3}} \frac{1}{\sqrt{3}} (\widehat{\imath} + \widehat{\jmath} + \widehat{\mathbf{k}}) \cdot (\widehat{\imath} - \widehat{\jmath} + \widehat{\mathbf{k}}) \\ &= \frac{2}{3}(1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1) \\ &= \mathbf{\frac{2}{3}} \end{aligned}$$

Question 19 Show that $\rho(x) = x$ is an integrating factor of

$$(3x + 2y^2)\,dx + 2xy\,dy = 0$$

and then solve the equation using the integrating factor.

Answer 19 The equation is not exact as written, since

$$\frac{\partial}{\partial y}(3x + 2y^2) = 4y \neq 2y = \frac{\partial}{\partial x}(2xy).$$

However if we multiply through by x, we get

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0,$$

which is exact since

$$\frac{\partial}{\partial y}(3x^2 + 2xy^2) = 4xy = 4xy = \frac{\partial}{\partial x}(2x^2y).$$

So we seek a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2$$

and

$$\frac{\partial f}{\partial y} = 2x^2y$$

The first of these equations can be integrated with respect to x to obtain

$$f(x, y) = x^3 + x^2y^2 + k(y)$$

for some unknown function $k(y)$. Now if we plug into the second equation, we get

$$2x^2y + k'(y) = 2x^2y,$$

showing that $k'(y) = 0$ and thus that $k(y)$ is constant.

So the solution is $f(x, y) = C$, or

$$\boldsymbol{x^3 + x^2y^2 = C}$$

Question 20 Find the distance from the point $A = (1, 0, 2)$ to the plane passing through the point $(1, -2, 1)$ and perpendicular to the line given by the parametric equations $x = 7$, $y = 1 + 2t$, $z = t - 3$.

Answer 20 The quick way is to use the projection formula, which I won't.

The long way is to actually find the point on the plane which is closest to A. To do this, we note that the normal vector to the plane is $\vec{n} = 2\widehat{\mathbf{j}} + \widehat{\mathbf{k}}$. So the equation of the plane is

$$0(x - 1) + 2(y + 2) + 1(z - 1) = 0$$

or $2y + z + 3 = 0$.

To get the closest point on the plane, we consider the line which is parallel to the normal vector on the plane and passes through A. This line has equation $x = 1$, $y = 2t$, $z = 2 + t$. The line intersects the plane $2y + z + 3 = 0$ when $4t + 2 + t + 3 = 0$, or $t = -1$. This occurs at the point $B = (1, -2, 1)$.

So the problem is to find the distance from A to B, which is

$$|\overrightarrow{AB}| = \sqrt{(1 - 1)^2 + (0 - (-2))^2 + (2 - 1)^2} = \sqrt{5}$$