Brief answers to Homework #10

Question 1

Section 13.3, #4

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \frac{\pi}{2}$$

Section 13.3, #11

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_0^{\ln 2} re^r\,dr\,d\theta = \pi(\ln 2 - \tfrac{1}{2})$$

Section 13.3, #14 The circle $x^2+(y-1)^2=1$ in polar coordinates is $r = 2\sin\theta$. So

$$\int_0^2\int_{-\sqrt{1-(y-1)^2}}^{0} xy^2\,dxdy = \int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r^4\cos\theta\sin^2\theta\,dr\,d\theta = -\frac{4}{5}$$

Section 13.3, #20

$$A = \int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta = \frac{64\pi^3}{27}$$

Section 13.3, #33 In polar coordinates, the function is $f(r,\theta) = \frac{1}{r}\ln r$ and the region is $1 \le r \le \sqrt{e}$. So

$$\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_1^{\sqrt{e}} \ln r\,dr\,d\theta = \pi(2-\sqrt{e})$$

(The answer in the textbook is wrong.)

Section 13.3, #44

(a) The Cartesian region is the triangle under the line $y = x/2$, above the x-axis, to the left of $x = 1$, and to the right of the origin.

(b) $y = x/2$ is given by $\theta = \arctan\frac{1}{2}$, $x = 1$ is given by $r = \sec\theta$, and $y = 0$ is given by $\theta = 0$.

(c) A plot in the $r\theta$-plane would just be the graph of $r = \sec\theta$, from $\theta = 0$ to $\theta = \arctan\frac{1}{2}$. (Note that in the $r\theta$-plane, r is along the y-axis and θ is along the x-axis. Thus the graph of this region does not resemble the graph of the actual region in the xy-plane.)

(d) The integral is

$$\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2}\, dy\, dx = \int_0^{\arctan(1/2)} \int_0^{\sec\theta} \cos\theta\, dr\, d\theta = \arctan\frac{1}{2}$$

Question 2

(a) $(1 + i)(3 - 2i) = 5 + i$

(b) $\dfrac{2 + i}{i} = 1 - 2i$

(c) $\dfrac{1 - 2i}{2 + i} = -i$

Question 3 $r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4) = (r + i)(r - i)(r + 2i)(r - 2i) = 0$, so the solutions are $r = i, -i, 2i, -2i$.

Question 4

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

Question 5

Section A.3, #2

(a) $x = \frac{5}{3}$, $y = -24$.

(b) $x = \frac{2}{5}$, $y = -\frac{1}{5}$.

(c) $x = -1$, $y = 0$.

Section A.3, #3 Write $z = re^{i\theta}$.

(a) $\overline{z} = re^{-i\theta}$, a reflection in the x-axis.

(b) $\overline{(-z)} = -re^{-i\theta} = re^{i(\pi-\theta)}$, a reflection in the y-axis.

(c) $-z = -re^{i\theta}$, a reflection through the origin.

(d) $1/z = 1/re^{-i\theta}$, so it is a reflection in the x-axis and a rescaling.

Section A.3, #7 $|z+1| = 1$ implies $|z+1|^2 = 1$, so

$$(x+1)^2 + y^2 = 1$$

This is a circle of radius 1, centered at $(-1, 0)$.

Section A.3, #8 $|z+1| = |z-1|$ implies $|z+1|^2 = |z-1|^2$, so

$$(x+1)^2 + y^2 = (x-1)^2 + y^2,$$

which simplifies to $x = 0$, a vertical line.

Section A.3, #12 $\dfrac{1+i}{1-i} = i = e^{\frac{i\pi}{2}}$

Section A.3, #24 Since $i = e^{i\pi/2}$, the two square roots are $\sqrt{i} = e^{i\pi/4}$ and $\sqrt{i} = e^{5i\pi/4}$.

Section A.3, #27 $z^4 - 2z^2 + 4 = 0$ implies

$$z^2 = 1 \pm \sqrt{3}i = 2e^{\pm i\pi/3}$$

So the four solutions are

$$z = \pm\sqrt{2}e^{\pm i\pi/6}$$

Question 6 $e^x \cos y = u$ and $e^x \sin y = v$ imply that $e^x = \sqrt{u^2 + v^2}$ and $\tan y = \frac{v}{u}$, so

$$x = \frac{1}{2}\ln\left(u^2 + v^2\right) \text{ and } y = \arctan\left(\frac{v}{u}\right)$$