

Brief answers to Homework #11

1. (Section 16.1, #6) If $y = \frac{C}{x} + \frac{x}{2}$, then $y' = -\frac{C}{x^2} + \frac{1}{2}$, so $y' + xy = x$.

2. (Section 16.1, #10) If $y = \frac{1}{x} \int_1^x \frac{e^t}{t} dt$, then by the fundamental theorem of calculus, $y' = -\frac{1}{x^2} \int_1^x \frac{e^t}{t} dt + \frac{1}{x} \frac{e^x}{x}$ so $x^2 y' + xy = e^x$

3. (Section 16.1, #15) The equation is separable.

$$\frac{dy}{y^2 + 1} = -\frac{dx}{x + 1} + \frac{dx}{x - 1}$$

Integrating both sides and using the initial condition $y(0) = 1$ gives

$$y = \tan \left(\ln \left| \frac{x - 1}{x + 1} \right| + \frac{\pi}{4} \right)$$

4. (Section 16.1, #24) The equation can be rewritten as

$$\frac{dy}{dx} = -\frac{1 + \left(\frac{y}{x}\right)^2}{\frac{y}{x}}$$

Since the right side depends only on $\frac{y}{x}$, the equation is homogeneous. Letting $y = vx$, we get $x \frac{dv}{dx} = -\frac{1 + 2v^2}{v}$. We solve this using separation and get

$$y = \pm x \sqrt{\frac{C}{x^4} - 1}$$

5. (Section 16.2, #4) If $M = x + y^2$ and $N = 2xy + 1$, then $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2y$, so the equation is exact.

6. (Section 16.2, #9) The equation is exact since $M_y = N_x$. The solution is

$$\frac{1}{2}x^2 + xy + \frac{1}{3}y^3 = C$$

7. (Section 16.2, #14) Rewriting the equation, we get

$$(y \cos xy + 1) dx + (x \cos xy + 2y - 3) dy = 0$$

This equation is exact. The solution is

$$\cos xy + x + y^2 - 3y = C$$

8. (Section 16.2, #19) After multiplying by $\rho = 1/y^2$, the equation becomes

$$(x + \frac{1}{y}) dx - \frac{x}{y^2} dy = 0$$

which is exact since $M_y = N_x = -\frac{1}{y^2}$. The solution is

$$\frac{1}{2}x^2 + \frac{x}{y} = C$$

9. (Section 16.2, #26) The equation is exact since $M_y = N_x = 1$. The general solution is

$$x^2 + xy + x + y^2 + y = C$$

With initial condition $y(1) = 5$, we get $C = 37$. Solve for y in terms of x using the quadratic formula, and we get

$$y = \frac{-x - 1 + \sqrt{(x+1)^2 - 4(x^2 + x - 37)}}{2}$$

10. (Section 16.3, #4) The solution of the complementary equation $xy' + y = 0$ is $y = \frac{C}{x}$. So $v = xy$ satisfies $v' = x \cos x$; thus $v = C + x \sin x +$

$\cos x$, and the solution is

$$y = \frac{C}{x} + \sin x + \frac{1}{x} \cos x$$

11. (Section 16.3, #7) The solution of $(\sin x)y' + (\cos x)y = 0$ is $y = \frac{C}{\sin x}$, so $v = y \sin x$ satisfies $v' = \tan x$, whose solution is $v = C - \ln |\cos x|$. Thus

$$y = \frac{C}{\sin x} - \frac{1}{\sin x} \ln |\cos x|$$

12. (Section 16.3, #9) The equation $y' + 2y = 0$ has general solution $y = Ce^{-2x}$. So $v = ye^{2x}$ satisfies $v' = xe^{2x}$, whose solution is $v = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$. Thus the general solution is $y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$. The initial condition implies $C = \frac{5}{4}$, so the solution is

$$y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}$$

13. (Section 16.3, #12) The solution of $xy' - 2y = 0$ is $y = Cx^2$. Letting $v = y/x^2$, we get $v' = \sec x \tan x$, whose solution is $v = C + \sec x$. So $y = Cx^2 + x^2 \sec x$. The initial condition implies $C = 2(\frac{9}{\pi^2} - 1)$. Thus the solution is

$$y = x^2 \left[2\left(\frac{9}{\pi^2} - 1\right) + \sec x \right]$$

14. (Section 16.3, #18) The solution is

$$x(t) = -10,000 + 11,000e^{-0.10t}.$$

The equation $x(t) = 100,000$ implies

$$t = 10 \ln 10 \approx 23 \text{ years}$$