Brief answers to Homework #12

Section 16.4, #4 The characteristic equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0$, so the solution is

$$y(x) = C_1 e^{3x} + C_2 e^{-x}$$

Section 16.4, #7 The characteristic equation is $r^2 - 10r + 25 = (r-5)^2 = 0$, so the solution is

$$y(x) = C_1 e^{5x} + C_2 x e^{5x}$$

Section 16.4, #12 The characteristic equation is $r^2 + 8r + 25 = 0$, with complex roots $r = -4 \pm 3i$, so the solution is

$$y(x) = C_1 e^{-4x} \cos(3x) + C_2 e^{-4x} \sin(3x)$$

Section 16.4, #14 The general solution is

$$y(x) = C_1 e^{x} + C_2 e^{-x/2}$$

The solution of the initial value problem is

$$y(x) = -\frac{1}{3} e^{x} - \frac{2}{3} e^{-x/2}$$

Section 16.4, #17 The general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

The solution of the initial value problem is

$$y(x) = x e^{-x}$$

Section 16.4, #20 The general solution is

$$y(x) = C_1 + C_2 x$$

The solution of the initial value problem is

$$y(x) = -3 + 5x$$

Section 16.5, #13 The solution of the homogeneous equation is

$$y_h(x) = C_1 e^{5x} + C_2 e^{-2x}$$

For the particular solution, we try $y_p(x) = A$. This gives $y_p'(x) = 0$ and $y_p''(x) = 0$, so $y_p(x) = \frac{3}{10}$. The general solution is

$$y(x) = C_1 e^{5x} + C_2 e^{-2x} + \frac{3}{10}$$

Section 16.5, #16 The solution of the homogeneous equation is

$$y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$$

For the particular solution, we try $y_p(x) = Ax^2 + Bx + C$. This gives $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Plugging in gives

$$2A + 4Ax + 2B + Ax^2 + Bx + C = x^2$$

which implies $A = 1$, $B = -4$, and $C = 6$. So the general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^2 - 4x + 6$$

Section 16.5, #21 The solution of the homogeneous equation is

$$y_h(x) = C_1 e^{x} + C_2 e^{-x}$$

For the particular solution, we try $y_p(x) = Axe^x + Bx^2 + Cx + D$. (We would use Ae^x but e^x is already a solution of the homogeneous equation.)

This gives $y_p'(x) = Axe^x + Ae^x + 2Bx + C$ and $y_p''(x) = Axe^x + 2Ae^x + 2B$. Plugging in gives

$$Axe^x + 2Ae^x + 2B - Axe^x - Bx^2 - Cx - D = e^x + x^2$$

which implies $A = \frac{1}{2}$, $B = -1$, $C = 0$, and $D = -2$. So the general solution is

$$y(x) = C_1e^x + C_2e^{-x} + \tfrac{1}{2}xe^x - x^2 - 2$$

Section 16.5, #26 The solution of the homogeneous equation is

$$y_h(x) = C_1e^x + C_2$$

For the particular solution, we try $y_p(x) = Ax^2+Bx$. (We would use $Ax+B$, but 1 is already a solution of the homogeneous equation.) Then $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Plugging in gives

$$2A - 2Ax - B = -8x + 3$$

which implies $A = 4$ and $B = 5$. So the general solution is

$$y(x) = C_1e^x + C_2 + 4x^2 + 5x$$

Section 16.5, #30 Plugging in gives

$$-A\cos x - B\sin x + A\sin x - B\cos x = \cos x + \sin x$$

which implies $A = 0$, $B = -1$, so $y_p(x) = -\sin x$. The solution of the homogeneous equation is

$$y_h(x) = C_1e^x + C_2$$

Thus the general solution is

$$y(x) = C_1e^x + C_2 - \sin x$$

Section 16.5, #44 The solution of the homogeneous equation is

$$y_h(x) = C_1e^x + C_2e^{2x}$$

So we try as a particular solution $y_p(x) = Axe^x + Bxe^{2x}$. (We would use $Ae^x + Be^{2x}$ but both e^x and e^{2x} are already solutions of the homogeneous equation.) Then $y_p'(x) = Axe^x + Ae^x + Be^{2x} + 2Bxe^{2x}$ and $y_p''(x) = Axe^x + 2Ae^x + 4Be^{2x} + 4Bxe^{2x}$. Plugging in gives

$$Axe^x + 2Ae^x + 4Be^{2x} + 4Bxe^{2x} - 3Axe^x - 3Ae^x - 3Be^{2x} - 6Bxe^{2x} \\ + 2Axe^x + 2Bxe^{2x} = e^x - e^{2x}$$

which implies $A = -1$ and $B = -1$. So the general solution is

$$y(x) = C_1e^x + C_2e^{2x} - xe^x - xe^{2x}$$

Section 16.5, #59 To solve $xy'' + 2y' = 1$, let $p(x) = y'(x)$. Then p satisfies the first-order equation

$$x\frac{dp}{dx} + 2p = 1$$

This is a linear differential equation and we can solve it using the standard method from Section 16.2.

First find the solution of the linear homogeneous equation

$$x\frac{dp}{dx} + 2p = 0$$

It is $p(x) = \frac{C}{x^2}$.

So we define a function $v(x)$ by $p(x) = \frac{v(x)}{x^2}$. Then $p'(x) = v'(x)/x^2 - 2v(x)/x^3$. Now plug this into $xp'(x) + 2p(x) = 1$, and get

$$\frac{1}{x}v'(x) - \frac{2v(x)}{x^2} + 2\frac{v(x)}{x^2} = 1$$

or simply $v'(x) = x$. This implies $v(x) = \frac{1}{2}x^2 + C$, which implies that

$$p(x) = \frac{v(x)}{x^2} = \frac{1}{2} + \frac{C}{x^2}$$

Finally, since $y'(x) = p(x)$, we have

$$y(x) = \int p(x)\,dx = \frac{1}{2}x - \frac{C_1}{x} + C_2$$

Now the initial conditions imply $2 = 1 - \frac{C_1}{2} + C_2$ and $1 = \frac{1}{2} + \frac{C_1}{4}$, so that $C_1 = 2$ and $C_2 = 2$. Thus

$$y(x) = \frac{1}{2}x - \frac{2}{x} + 2$$