Solutions to the Final Exam

Q 1. Consider the following utility game:

	C	N
C	$(-1, 2)$	$(0, 1)$
N	$(1, -1)$	$(-2, 0)$

(a) If Row doesn't know anything about Column's choice of strategy, what should Row's optimal strategy be?

Answer Let's say Row will pick C with probability Q, and will pick N with probability $(1 - Q)$. Row should consider the two extreme cases: either Column always chooses C, or always chooses N.

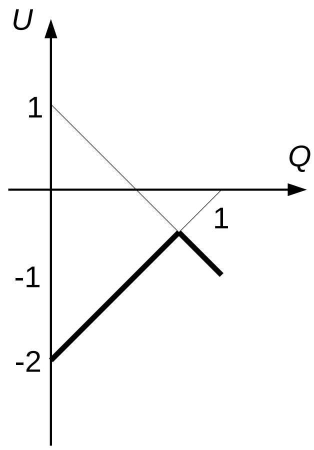
If Column always chooses C, then Row's expected utility will be

$$U = -1 \cdot Q + 1 \cdot (1 - Q) = 1 - 2Q.$$

If Column always chooses N, then Row's expected utility will be

$$U = 0 \cdot Q + -2 \cdot (1 - Q) = 2Q - 2.$$

We graph these two lines on the same axes, from $Q = 0$ to $Q = 1$.



The lines have opposite slopes and intersect. For the minimax strategy, we are concerned about the bottom parts of the graph (in bold here): we want to find the highest point on this bottom part.

To find Q, we find where the lines intersect.

$$\begin{aligned} 1 - 2Q &= 2Q - 2 \\ 3 &= 4Q \\ Q &= 3/4 \end{aligned}$$

So the optimal strategy for Row is to pick C with a probability of 75% and pick N with a probability of 25%.

(b) If Column knows that Row is using the strategy you found in part (a), what should Column's optimal strategy be?

Answer If Column always chooses C, Column will expect a utility of

$$U = 2 \cdot 0.75 - 1 \cdot 0.25 = 1.25.$$

If Column always chooses N, Column will expect a utility of

$$U = 1 \cdot 0.75 + 0 \cdot 0.25 = 0.75.$$

Any mixed strategy will yield a utility between these two. So Column should always pick C to obtain the highest utility.

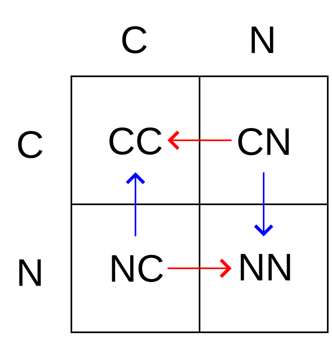
Q 2. Suppose that we have a two-by-two ordinal game, with options C and N for both Row and Column.

Suppose that neither player has a dominant strategy.

Prove that if CC is a Nash equilibrium, then NN is also a Nash equilibrium.

(Remember not to assume anything other than what's given! Prove this in general, not for a particular game.)

Answer



Since CC is a Nash equilibrium, Row must prefer CC to NC. If Row also preferred CN to NN, then Row would have a dominant strategy of C. Therefore Row must instead prefer NN to CN. (Refer to the diagram: if Row's blue preference arrow goes one way in one column, it must go the other way in the other column.)

Similarly, Column must prefer CC to CN, and in order not to have a dominant strategy of C, Column must prefer NN to NC.

Since both players prefer NN over the neighboring alternatives, NN is also a Nash equilibrium.

Q 3. Consider a model of negotiating for a car. The buyer will not pay more than $13,000. The dealer will not accept less than $10,000.

The buyer starts, bidding either $10k, $11k, $12k, or $13k. Then the dealer responds, either accepting the buyer's offer or proposing something higher (multiples of $1k, up to $13k). Next, the buyer either accepts the dealer's offer or makes a new bid, higher than her last offer but lower than the dealer's offer. Finally, the dealer either accepts the buyer's new bid or proposes a new offer, higher than the buyer's offer but lower than his previous offer. (With these numbers, this is the longest it can go.)

Construct a decision tree to model this situation. Of course the buyer always prefers paying the lowest price, while the dealer always prefers getting the highest price. In case of a tie (two bids resulting in the same ultimate outcome), both the buyer and the dealer will make the lower bid.

Prune the tree and determine what happens.

Answer In the decision tree, the first row (for the buyer) has either 10, 11, 12, or 13.

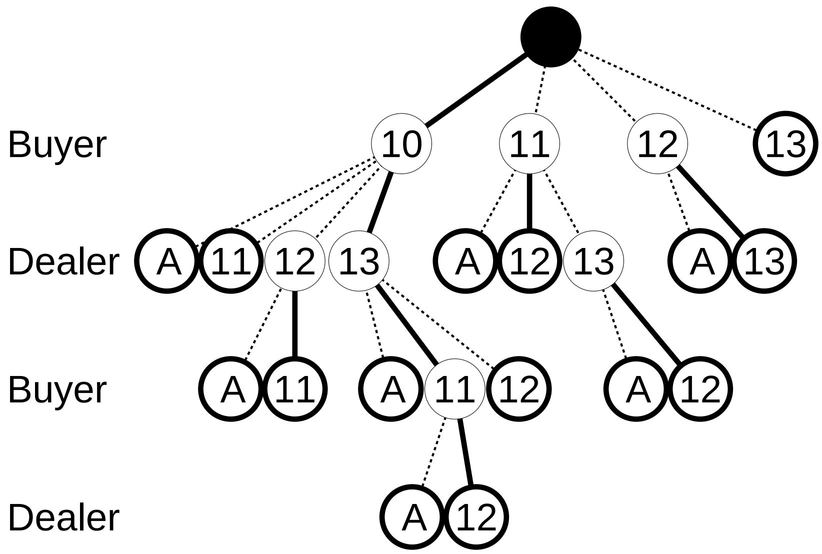
If the buyer bids 10, the dealer can either accept 10, or bid 11, 12, or 13. If the buyer bids 11, the dealer can either accept 11, or bid 12 or 13. If the buyer bids 12, the dealer can either accept 12 or bid 13. If the buyer bids 13, the dealer will accept 13.

If the buyer has bid 10 and the dealer has bid 11, then the buyer has no other option but to accept 11, and the game ends. Similarly whenever the dealer offers one more than the buyer, the buyer will accept the dealer's bid and the game ends. If the buyer has bid 10 and the dealer bid 12, then the buyer can either accept 12 or bid 11. If the buyer bids 11, the dealer must accept it.

If the buyer has bid 10 and the dealer has bid 13, then the buyer can bid either 11 or 12, or accept 13. If the buyer bids 12, the dealer must accept. If the buyer bids 11, then the dealer (in the next row) can accept or bid 12. If the dealer bids 12, the buyer must accept.

If the buyer has bid 11 and the dealer has bid 13, then the buyer can bid 12 and the dealer must accept.

Here is the decision tree.



Now we prune from the last row. The dealer can end the game either by accepting 11 or bidding 12, so he will bid 12.

Now the third row. In the left group, the buyer can either accept 12 or demand 11, so she will demand 11. In the middle group, the buyer can either accept 13 or bid 12 or bid 11 (which she knows will lead to 12 anyway). Since she chooses the lowest bid in case of a tie, she will pick 11. In the right group, the buyer will bid 12 rather than accepting 13.

Now the second row. In the left group, the dealer knows that accepting 10 will give him 10; bidding 11 will give him 11; bidding 12 will also give him 11; and bidding 13 will eventually give him 12. So he will bid 13. In the middle row, the dealer knows that accepting 11 will give him 11; bidding 12 will give him 12; and bidding 13 will eventually give him 12. Since he chooses the cheapest option that gives him 12, he will bid 12. In the right group, the dealer knows that accepting 12 will give him 12, and bidding 13 will give him 13. So he will bid 13.

Now the first row. The buyer now knows that bidding 10 or 11 will mean a price of 12, and that bidding 12 or 13 will mean a price of 13. So the buyer will choose to bid 10.

The ultimate outcome is that the buyer bids 10, the dealer offers 13, the buyer bids 11, and the dealer offers 12, where they stop.

Q 4. Consider the Awkward Social Interaction game.

In this game, a professor and a student pass each other in the street. Each one has a choice to either smile and say, “Hello” (S), or to ignore the other (I).

Their primary preference is to both do the same thing. The professor (an antisocial type) would prefer that both ignore each other, while the student would prefer that both smile. Each one thinks the worst outcome would be to smile and be ignored by the other.

Fill in the matrix with the preferences. (Here the student is Row and the professor is Column.)

Answer The primary preference means that both the student and professor have their top preferences (3 and 4) in the upper left corner and the upper right corner. The professor's 4 is in the lower right, while the student's 4 is the upper left.

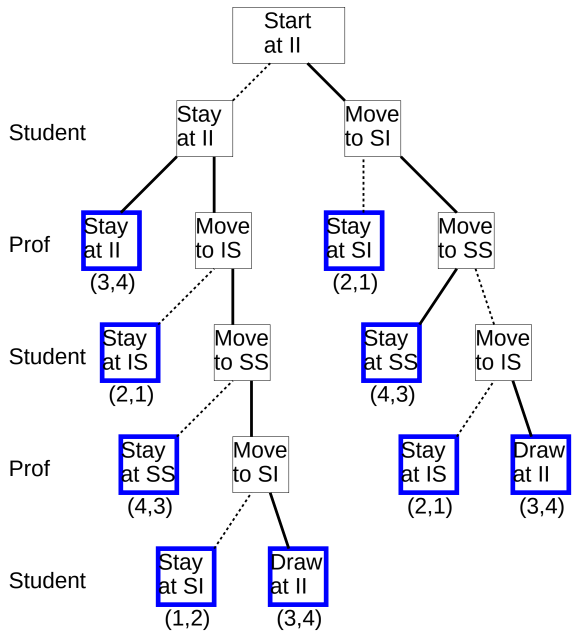
The 1 for the professor is in IS, so the 2 for professor must be in SI. The 1 for the student is SI, so 2 for the student is IS.

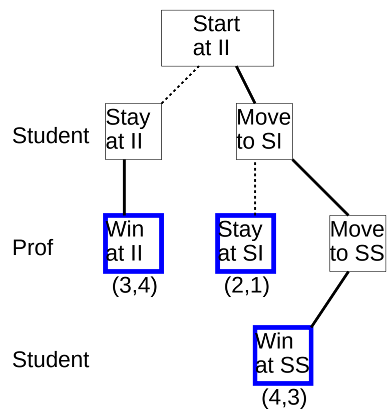
	S	I
S	(4,3)	(1,2)
I	(2,1)	(3,4)

Suppose they can change their expressions freely based on what the other is doing. Determine whether mutual ignoring is a non-myopic equilibrium when the student goes first.

Answer We can do this two equivalent ways. On the left we see the long version of the decision tree, which incorporates the draw criterion but not the win criterion. In this one, if the student stays at II, the professor can either stay there or move; if he moves, they will go all the way around the board and end up back at II. Thus the professor gets II whether he stays or moves, and therefore has no real preference between the two. The student therefore decides to move to SI, which forces the professor to move as well to SS, and then the student chooses to stay at SS.

On the right the decision tree is much shorter, since we incorporate the fact that if the student stays at II, it's a win for the professor (who gets a 4 and will stop there); and if the professor moves to SS, it's a win for the student (who gets a 4 and will stop there). The short version also predicts that the student will move to SI and then the professor will move to SS.





Either way, they choose not to stay at II, and therefore II is not a non-myopic equilibrium.

Q 5.

(a) Suppose we held a dollar-auction in class: students could bid for \$10 in increments of \$1. If you knew in advance that I would stop the auction at \$50, what would O'Neill's Theorem tell you is the optimal bid?

Answer In this case $b = 50$ and $s = 10$, so that $(s - 1) = 9$. Subtracting multiples of 9 from 50, we get the numbers 41, 32, 23, 14, and 5. So the optimal opening bid is \$5.

(b) If you got to make the first bid, why might you not actually bid this in the real situation? Explain in a sentence or two.

Answer There are several possible reasons. O'Neill's theorem assumes perfect rationality, i.e., that everyone can see 50 steps ahead. In the real situation, if at least one person in the class couldn't see that far ahead, that person might be tempted to bid \$9 or something else. If even one person does this, you'll lose money.

Alternatively, if the other students can't see that many steps ahead, they might be afraid to bid at all, in which case you could bid less than \$5 and win even more.

In addition, O'Neill's theorem assumes that each player is operating with the conservative convention, which typically doesn't happen in class (if someone bids against you and makes you lose money, you'll try to punish him/her). If the punishing convention is used, it turns out that there is no optimal opening bid (in particular, O'Neill's theorem does not apply).

Finally, it assumes that each player knows the rule that the other will use to break ties; in the real situation, you couldn't know what rules the other students were using.

(c) To what extent do you think O'Neill's Theorem is useful for understanding real-world escalation situations? Explain your answer in one or two sentences.

Answer O'Neill's theorem itself applies in such a limited context that it is not useful in many situations. It would be most useful in a situation where the bankroll is very small (so that both sides could see all the way to the end, and thus the perfect rationality assumption was justified), the bankroll and stakes were the same for both sides, and the two sides were basically magnanimous with each other (so that the conservative convention could be assumed). In these cases, O'Neill's theorem would give an accurate prediction.

For the most part, real escalation situations could not be understood using O'Neill's theorem. The reasons are many: the bankrolls are almost never the same; the stakes are often not the same; the bidding increments are not necessarily the same, and need not be integers.

Most importantly, even if an escalation situation could be modeled strictly monetarily, the assumption of long-term rationality on both sides is usually not justified. In fact, the original purpose of the dollar auction was to demonstrate why people behave irrationally in the long term while pursuing short-term benefits, or preventing short-term losses.

The main usefulness of O'Neill's theorem is as a theoretical tool, not so much in the specific answer it gives; rather in the fact that it gives an example of how one could model a very large, complex, rigidly structured game so as to find an optimal strategy without using brute force.

For parts (b) and (c), you got full credit if you gave at least one of the many possible answers written above.