Solutions to Final Exam

Q 1. Each of the matrices below is a row echelon form of the augmented matrix of a system of linear equations. Determine if each system is consistent or inconsistent and briefly justify your answer. If the system is consistent, find the solution.

(a) $$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Answer The system is consistent. The general solution is $x_2 = 3, x_1 = -3 - 4x_3$, where x_3 is arbitrary.

(b) $$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 3 & 3 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Answer The system is inconsistent since $\operatorname{rank}(A) = 2$ but $\operatorname{rank}(A|b) = 3$.

(c) $$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Answer The system is inconsistent since the matrix reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 1 & -6/7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

For this matrix, $\operatorname{rank}(A) = 3$ but $\operatorname{rank}(A|b) = 4$.

Q 2. Solve $\mathbf{X}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ subject to $\mathbf{X}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hint: There is a particular solution of the form $X_p(t) = \begin{bmatrix} A \\ B \end{bmatrix} e^{-t}$.

Answer First we find the particular solution, given the trial form. We obtain the equation

$$\begin{bmatrix}-A\\-B\end{bmatrix}e^{-t}=\begin{bmatrix}0&2\\-1&3\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}e^{-t}+\begin{bmatrix}1\\-1\end{bmatrix}e^{-t}.$$

This yields the two equations $-A=2B+1$ and $-B=-A+3B-1$, which have solutions $A=-1$ and $B=0$. So

$$X_p(t)=\begin{bmatrix}-1\\0\end{bmatrix}e^{-t}.$$

Next we find the complementary solution of $\mathbf{X}'=\begin{bmatrix}0&2\\-1&3\end{bmatrix}\mathbf{X}$. The characteristic equation of A is $\lambda^2-3\lambda+2=0$, with solutions $\lambda=1$ and $\lambda=2$. For $\lambda=1$, the eigenvector is $\begin{bmatrix}2\\1\end{bmatrix}$, while for $\lambda=2$, the eigenvector is $\begin{bmatrix}1\\1\end{bmatrix}$. Therefore the complementary solution is

$$X_c(t)=c_1\begin{bmatrix}2\\1\end{bmatrix}e^{t}+c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t}.$$

So the general solution is

$$X(t)=c_1\begin{bmatrix}2\\1\end{bmatrix}e^{t}+c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t}+\begin{bmatrix}-1\\0\end{bmatrix}e^{-t}.$$

The solution with initial condition $\mathbf{X}(0)=\begin{bmatrix}0\\0\end{bmatrix}$ has $2c_1+c_2-1=0$ and $c_1+c_2=0$, so that $c_1=1$ and $c_2=-1$. Thus the solution is

$$X(t)=\begin{bmatrix}2\\1\end{bmatrix}e^{t}-\begin{bmatrix}1\\1\end{bmatrix}e^{2t}+\begin{bmatrix}-1\\0\end{bmatrix}e^{-t}=\begin{bmatrix}2e^{t}-e^{2t}-e^{-t}\\e^{t}-e^{2t}\end{bmatrix}.$$

Q 3.

(a) Show that the vectors $\mathbf{u}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\mathbf{u}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$ are linearly independent.

Answer We can either solve the equations $a_1\mathbf{u}_1+a_2\mathbf{u}_2+a_3\mathbf{u}_3=\mathbf{0}$ and show that a_1, a_2, and a_3 all have to be zero, or we can simply compute the determinant of the matrix P formed by the column vectors and show that it is nonzero:

$$\begin{vmatrix}1&0&1\\1&1&1\\1&1&0\end{vmatrix}=0+0+1-1-1-0=-1\neq 0.$$

(b) Express the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Answer The equation $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{v}$ implies the three equations

$$\begin{array}{ccccccc} a_1 & + & & + & a_3 & = & 3 \\ a_1 & + & a_2 & + & a_3 & = & 2 \\ a_1 & + & a_2 & & & = & 1, \end{array}$$

which implies that $a_2 = -1$, $a_1 = 2$, and $a_3 = 1$. Thus

$$\mathbf{v} = 2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3.$$

Q 4. Use Laplace Transforms to solve $y'' - 4y' = 6e^{3t}$, $y(0) = 1$, $y'(0) = -1$.

Answer We have $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$, and $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1$. Therefore the equation is

$$s^2Y(s) - s + 1 - 4sY(s) + 4 = \frac{6}{s-3},$$

which simplifies to

$$(s^2 - 4s)Y(s) = s - 5 + \frac{6}{s-3} = \frac{s^2 - 8s + 21}{s-3}.$$

So

$$Y(s) = \frac{s^2 - 8s + 21}{s(s-3)(s-4)}.$$

Expanding $Y(s)$ in partial fractions, we have

$$Y(s) = \frac{s^2 - 8s + 21}{s(s-3)(s-4)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-4}$$

$$s^2 - 8s + 21 = A(s-3)(s-4) + Bs(s-4) + Cs(s-3).$$

When $s = 0$, we find $A = \frac{7}{4}$. When $s = 3$, we find $B = -2$. When $s = 4$, we find $C = \frac{5}{4}$. Thus

$$Y(s) = \frac{7}{4}\frac{1}{s} - 2\frac{1}{s-3} + \frac{5}{4}\frac{1}{s-4}.$$

The inverse transform of $\frac{1}{s-a}$ is e^{at}, so that

$$y(t) = \frac{7}{4} - 2e^{3t} + \frac{5}{4}e^{4t}.$$

Q 5. Use diagonalization to solve the system

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{X}.$$

Answer The characteristic equation for A is $\lambda^2 - 2 = 0$, so the eigenvalues are $\lambda = \pm\sqrt{2}$. The eigenvectors are $\mathbf{K}_1 = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$ for $\sqrt{2}$ and $\mathbf{K}_2 = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ for $-\sqrt{2}$. Thus we can write

$$P = \begin{bmatrix} 1 & \sqrt{2}+1 \\ \sqrt{2}-1 & 1 \end{bmatrix}.$$

So the general solution is

$$\mathbf{X} = \begin{bmatrix} 1 & \sqrt{2}+1 \\ \sqrt{2}-1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{\sqrt{2}t} \\ c_2 e^{-\sqrt{2}t} \end{bmatrix} = \begin{bmatrix} c_1 e^{\sqrt{2}t} + c_2(\sqrt{2}+1)e^{-\sqrt{2}t} \\ c_1(\sqrt{2}-1)e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} \end{bmatrix}.$$

(There are several possible forms of this solution, depending what multiples of the eigenvectors you chose.)

Q 6. Decide if $x = 0$ is an ordinary point, a regular or an irregular singularity for the following ODE: $(e^x - 1 - x)y'' + xy = 0$. Briefly justify your decision. *DO NOT SOLVE THE EQUATION!*

Hint: the Maclaurin series for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

Answer In standard form, the differential equation is

$$y'' + \frac{x}{e^x - 1 - x} y = 0.$$

The coefficient $Q(x)$ looks like

$$Q(x) = \frac{x}{e^x - 1 - x} = \frac{x}{x^2/2 + x^3/6 + \cdots} = \frac{1}{x/2 + x^2/6 + \cdots}.$$

At $x = 0$, this function is singular. Therefore $x = 0$ is a singularity.

To determine if it is a regular singularity, we check to see whether $x^2 Q(x)$ is analytic at $x = 0$. We have

$$x^2 Q(x) = \frac{x^2}{x/2 + x^2/6 + \cdots} + \frac{x}{1/2 + x/6 + \cdots}.$$

The numerator is x and the denominator is a power series which is analytic and nonzero at $x = 0$. Since we can (at least theoretically) divide one power series by a nonzero power series to get another power series, we see that $x^2 Q(x)$ is analytic at $x = 0$. So $x = 0$ is a regular singularity.

Q 7. Use an appropriate power series method about $x = 0$ to find the two solutions of the given ODE.

$$2xy'' + y' + y = 0.$$

For each solution, find a recursion relation for the coefficients, and write the first three terms of the series.

Answer The equation is singular at $x = 0$, but the singularity is regular, so we can use the method of Frobenius. Therefore we try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We then obtain

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} = \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)x^{n+r}$$

$$2xy''(x) = 2x \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} = \sum_{n=-1}^{\infty} 2a_{n+1}(n+r+1)(n+r)x^{n+r}.$$

Plugging in, we obtain

$$2a_0 r(r-1) + a_0 r + \sum_{n=0}^{\infty} \Big[a_{n+1}(n+r+1)(2n+2r+1) + a_n \Big] x^{n+r} = 0.$$

The indicial equation is thus $2r(r-1) + r = 2r^2 - r = 0$, with solutions $r = 0$ and $r = \frac{1}{2}$.

If $r = 0$, we get the recursion relation

$$a_{n+1} = -\frac{1}{(n+1)(2n+1)} a_n,$$

and the first few terms of the solution with $a_0 = 1$ are

$$y_1(x) = 1 - x + \frac{1}{6}x^2 + \cdots .$$

If $r = 1/2$, we get the recursion relation

$$a_{n+1} = -\frac{1}{(n+1)(2n+3)} a_n,$$

and the first few terms with $a_0 = 1$ are

$$y_2(x) = x^{1/2}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 + \cdots\right).$$

Q 8. Use Green's Theorem to evaluate $\oint_C y^2\,dx + x^2\,dy$ where C is the triangle bounded by $x = 0$, $x + y = 1$, and $y = 0$.

Answer There are no singularities, so Green's Theorem says

$$\begin{aligned}
\oint_C y^2\,dx + x^2\,dy &= \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right)\,dA \\
&= \iint_R (2x - 2y)\,dA \\
&= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx \\
&= \int_0^1 \left(2x(1-x) - (1-x)^2\right)dx \\
&= \int_0^1 (4x - 1 - 3x^2)\,dx \\
&= 2 - 1 - 1 \\
&= 0.
\end{aligned}$$

Q 9. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F} = xz\,\mathbf{i} + xy\,\mathbf{j} + 3xz\,\mathbf{k}$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant traversed counterclockwise as viewed from above.

Answer Since there are no singularities, Stokes' Theorem says

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS.$$

If the surface is given by $z = g(x, y)$, then

$$\mathbf{n}\,dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)\,dx\,dy.$$

In this case, $z = 2 - 2x - y$, so that

$$\mathbf{n}\,dS = (2\mathbf{i} + \mathbf{j} + \mathbf{k})\,dx\,dy.$$

We compute

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = 0\,\mathbf{i} - (3z - x)\,\mathbf{j} + y\,\mathbf{k} = (x - 3z)\,\mathbf{j} + y\,\mathbf{k}.$$

The projection of the surface in the xy-plane is the triangle bounded above by $y = 2 - 2x$, below by $y = 0$, and between $x = 0$ and $x = 1$. Thus the surface integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS \\ &= \int_0^1 \int_0^{2-2x} (x + y - 3z)\, dy\, dx \\ &= \int_0^1 \int_0^{2-2x} (7x + 4y - 6)\, dy\, dx \\ &= \int_0^1 (7xy + 2y^2 - 6y)\Big|_0^{2-2x}\, dx \\ &= \int_0^1 (14x - 14x^2 + 8 - 16x + 8x^2 - 12 + 12x)\, dx \\ &= \int_0^1 (10x - 6x^2 - 4)\, dx \\ &= (5x^2 - 2x^3 - 4x)\Big|_0^1 \\ &= -1. \end{aligned}$$

Q 10. Show that the given integral is path independent, then evaluate the integral two ways: (a) find a function $d\phi = P\,dx + Q\,dy$ and, (b) integrate along *any* convenient path between the given points.

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y\, dx + (1 - \sin x \sin y)\, dy$$

Answer To prove it's path independent, we compute

$$\frac{\partial P}{\partial y} = -\cos x \sin y = \frac{\partial Q}{\partial x}.$$

Because the integral is path independent, we can solve the equations

$$\frac{\partial \phi}{\partial x} = \cos x \cos y \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = 1 - \sin x \sin y.$$

We obtain $\phi(x, y) = \sin x \cos y + y$. Thus, for part (a), the integral is

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y \, dx + (1 - \sin x \sin y) \, dy = \phi\left(\tfrac{\pi}{2}, 0\right) - \phi(0, 0) = 1.$$

For part (b), we construct the simplest path, a straight line: $x = t$, $y = 0$, from $t = 0$ to $t = \pi/2$. Then $dx = dt$ and $dy = 0$. The integral is

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y \, dx + (1 - \sin x \sin y) \, dy = \int_0^{\pi/2} \cos t \, dt = \sin t \Big|_0^{\pi/2} = 1.$$

Q 11. Solve $x^2 y'' - 7xy' + 41y = 0$.

Answer This is just a Cauchy-Euler equation, and we can find solutions of the form $y = x^r$. Plugging in $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$, we obtain the characteristic equation

$$0 = r(r-1) - 7r + 41 = r^2 - 8r + 41 = (r-4)^2 + 25 = 0.$$

The solutions are $r = 4 \pm 5i$. Thus the general solution will be

$$y = c_1 x^4 \cos(5 \ln x) + c_2 x^4 \sin(5 \ln x).$$

Q 12. Evaluate the surface integral $\iint_S G(x, y, z) \, dS$ if $G(x, y, z) = x + y + z$ and S is the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$.

Answer We cannot use any shortcuts here, since the surface is not closed. Thus we compute directly:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

$$= \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} \, dA = \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} \, dA = \sqrt{2} \, dA.$$

The region of integration is the annulus enclosed by the circles $r = 1$ and $r = 4$. Thus the integral is most natural in polar coordinates, and the function to integrate is $G(x, y, z) = x + y + z = r \cos\theta + r \sin\theta + r$.

We thus have

$$\begin{aligned}
\iint_S G(x,y,z)\,dS &= \iint_R G(x,y,\sqrt{x^2+y^2})\,\sqrt{2}\,dA \\
&= \sqrt{2}\int_0^{2\pi}\int_1^4 (r\cos\theta + r\sin\theta + r)\,r\,dr\,d\theta \\
&= 2\pi\sqrt{2}\int_1^4 r^2\,dr \\
&= 2\sqrt{2}\pi\frac{1}{3}(4^3-1^3) \\
&= 42\sqrt{2}\pi.
\end{aligned}$$