Solutions to Final Exam

Q 1. TRUE or FALSE.

(a) The radius of convergence of the Taylor series of $f(z) = \dfrac{\sin z}{(z+2)^2}$ around $z = 1$ is $R = 1$.

Answer This is false. The radius of convergence is the distance to the nearest isolated singularity, which occurs at $z = -2$. Thus $R = |1 - (-2)| = 3$.

(b) Let C_1 be the positively oriented circle of radius 2 and let C_2 be the positively oriented circle of radius 5. Then

$$\oint_{C_1} \left[\frac{1}{z+1} + \frac{1}{z-4} \right] dz = \oint_{C_2} \left[\frac{1}{z+1} + \frac{1}{z-4} \right] dz$$

Answer This is false. The left side is $2\pi i$ since the contour only encloses the singularity at $z = -1$, while the right side is $4\pi i$ since the contour encloses both singularities.

Q 2.

(I) If $f(x) = x^2 - x$ is expanded in a Fourier series on the interval $[-2, 2]$, then at $x = 2$ the series will converge to

(a) 0
(b) 1
(a) 2
(c) 3
(d) 4
(e) 5
(f) 6

Answer The periodic extension of $f(x)$ will have a discontinuity at $x = 2$. From the left it will approach $f(2) = 2$. From the right it will approach $f(-2) = 6$. Thus the Fourier series will converge to 4, the average of the two, which is (d).

(II) What is the image of the line $x = 1$ under the map $f(z) = e^z$?

(a) A circle
(b) A horizontal line
(c) A vertical line
(d) A hyperbola
(e) A logarithmic spiral
(f) None of these

Answer If $x = 1$, then $z = 1 + iy$ for some real number y, and thus $f(z) = e^{1+iy} = e(\cos y + i \sin y)$. This is the parametric equation of a circle of radius e centered at the origin. Thus the answer is (a).

Q 3. Compute the complex Fourier series for $f(x) = \begin{cases} 3, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ which is valid on the interval $(-2, 2)$.

Answer From the general formula:

$$c_n = \frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}\,dx = \frac{1}{4}\int_{-1}^{1} 3e^{-in\pi x/2}\,dx$$
$$= \frac{3}{4}\,\frac{2}{-in\pi}\left(e^{-in\pi/2} - e^{in\pi/2}\right) = \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

This doesn't work for $n = 0$, so we have to do a separate computation:

$$c_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} 3\,dx = \frac{3}{2}.$$

The complex Fourier series is thus

$$f(x) = \frac{3}{2} + \sum_{n \neq 0}^{\infty} \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)e^{in\pi x/2}$$

Q 4. Using separation of variables find the solution of

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

satisfying

$$u(0,t) = 0, \qquad u\left(\frac{\pi}{2}, t\right) = 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = f(x)$$

Express the coefficients of the solution in terms of $f(x)$.

Answer Try a solution of the form $u(x,t) = X(x)T(t)$. Then we get

$$\frac{T''(t)}{T(t)} - 2\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2.$$

(Since this is one of the standard Sturm-Liouville problems for $X(x)$, we can assume the eigenvalue is of the form $-\lambda^2$ to save time.)

We get $X(x) = A\cos\lambda x + B\sin\lambda x$, and the conditions $X(0) = 0$ and $X(\pi/2) = 0$ imply that $A = 0$ and $\lambda = 2n$ for some positive integer n.

Then the equation for $T(t)$ is

$$T''(t) - 2T'(t) + 4n^2 T(t) = 0,$$

which has characteristic equation $r^2 - 2r + 4n^2 = 0$, with solutions $r = 1 \pm \sqrt{1-4n^2} = 1 \pm i\sqrt{4n^2-1}$. Thus the solution is $T(t) = Ce^t\cos(\sqrt{4n^2-1}t) + De^t\sin\sqrt{(4n^2-1t)}$. With the initial condition $T(0) = 0$, we see that $C = 0$.

Then the general solution can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^t \sin(\sqrt{4n^2-1}t)\sin 2nx$$

The initial condition can be expressed as

$$u_t(x,0) = \sum_{n=1}^{\infty} a_n\sqrt{4n^2-1}\sin 2nx = f(x),$$

so that the formula for a_n is

$$a_n = \frac{4}{\pi\sqrt{4n^2-1}}\int_0^{\pi/2} f(x)\sin 2nx\,dx$$

Q 5. Find the steady-state temperature in a semicircular plate of radius 2, if the round part is held at a temperature of 100 degrees and the flat bottom part is held at 0 degrees.

Answer We are solving the Laplace equation in polar coordinates, so using separation of variables and assuming $u(r,\theta) = R(r)\Theta(\theta)$, we get

$$\frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2$$

where again we know that the eigenvalue must be negative for a standard Sturm-Liouville problem.

For Θ we get the solution $\Theta(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$. The boundary conditions translate into $\Theta(0) = 0$ and $\Theta(\pi) = 0$, so that $A = 0$ and $\lambda = n$ for some positive integer n.

For R we then get the equation $r^2R''(r) + rR'(r) - n^2R(r) = 0$, which is well-known to have solutions $R(r) = Cr^n + Dr^{-n}$. Since at $r = 0$ the temperature must be finite, we exclude the solution r^{-n} and obtain only r^n.

Now we write the general solution as a summation:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin n\theta.$$

To satisfy the boundary condition $u(2,\theta) = 100$, we must have

$$2^n a_n = \frac{2}{\pi}\int_0^{\pi} 100\sin n\theta\, d\theta = \frac{200}{n\pi}\big(1 - (-1)^n\big).$$

So the solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{200\big(1-(-1)^n\big)}{n\pi}\left(\frac{r}{2}\right)^n \sin n\theta.$$

Q 6. Using a Fourier Transform find the steady-state temperature $u(x,y)$ on the semi-infinite plate satisfying

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \pi, y > 0 \\ u_y(x,0) &= 0, \qquad 0 < x < \pi \\ u(0,y) &= 0, \qquad u(\pi,y) = \begin{cases} 1 & y < 1 \\ 0 & y > 1 \end{cases} \end{aligned}$$

Express your final answer as an inverse Fourier transform integral.

Answer The variable with infinite range is y, which goes from 0 to ∞. Thus we must use either a Fourier sine or Fourier cosine transform. The condition $u_y(x,0) = 0$ tells us to use the cosine transform. So let

$$U(x,\alpha) = \mathcal{F}_c\{u(x,y)\} = \int_0^{\infty} u(x,y)\cos\alpha y\, dy.$$

Then

$$\mathcal{F}_c\{u_{xx}(x,y)\} = U_{xx}(x,\alpha)$$
$$\mathcal{F}_c\{u_{yy}(x,y)\} = -\alpha^2 U(x,\alpha) - u_y(x,0) = -\alpha^2 U(x,\alpha),$$

so the Laplace equation becomes

$$U''(x) - \alpha^2 U(x) = 0,$$

which has solution $U(x,\alpha) = A\cosh\alpha x + B\sinh\alpha x$.

To find A and B, we transform the boundary conditions. We have $U(0,\alpha) = \mathcal{F}_c\{u(0,y)\} = 0$ and

$$U(\pi,\alpha) = \mathcal{F}_c\{u(\pi,y)\} = \int_0^1 \cos\alpha y\,dy = \frac{\sin\alpha}{\alpha}.$$

Thus $A = 0$ and $B\sinh\alpha\pi = \dfrac{\sin\alpha}{\alpha}$.

Performing the inverse transform, we get

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh\alpha x}{\sinh\alpha\pi}\,\frac{\sin\alpha}{\alpha}\,\cos\alpha y\,dy$$

Q 7. Find the Laurent series of $f(z) = \dfrac{2z}{z^2-2z-8}$ valid for $|z+2| > 6$.

Answer We first convert into partial fractions and get

$$f(z) = \frac{2}{3}\frac{1}{z+2} + \frac{4}{3}\frac{1}{z-4}.$$

The first term is already in a Laurent series centered at $z=-2$, so we only need to work on the second term. We have

$$\frac{1}{z-4} = \frac{1}{z+2-6} = \frac{1}{z+2}\frac{1}{1-\frac{6}{z+2}} = \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{6}{z+2}\right)^n = \sum_{n=0}^{\infty}\frac{6^n}{(z+2)^{n+1}}.$$

Thus the full Laurent series in $|z+2| > 6$ for the original function is

$$f(z) = \frac{2}{3}\frac{1}{z+2} + \frac{4}{3}\sum_{n=0}^{\infty}\frac{6^n}{(z+2)^{n+1}}$$

Q 8. Classify the singularity of each of the following functions at $z_0 = 0$ and find the corresponding residue.

(a) $f(z) = \dfrac{1}{z(e^z - 1)}$

Answer Near $z = 0$, we have the approximation $e^z - 1 \approx z$, so that $f(z) \approx \frac{1}{z^2}$. This tells us that $z_0 = 0$ is a pole of order 2. To compute the residue, we use

$$\begin{aligned}\operatorname*{Res}_{z=0} \frac{1}{z(e^z-1)} &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{e^z-1}\right) = \lim_{z\to 0}\frac{(e^z-1)(1)-(z)(e^z)}{(e^z-1)^2} \\ &= \lim_{z\to 0}\frac{e^z-e^z-ze^z}{2(e^z-1)e^z} = \lim_{z\to 0}\frac{-z}{2(e^z-1)} = \lim_{z\to 0}\frac{-1}{2e^z} = -\frac{1}{2}\end{aligned}$$

where we used L'Hôpital's Rule twice to compute the 0/0 limit at the end.

(b) $f(z) = z^3 e^{-1/z^2}$

Answer To find the nature of the singularity, we need to expand in a full Laurent series. We start with the known Maclaurin series for the exponential:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Thus

$$f(z) = z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n! z^{2n}} = z^3 - z + \frac{1}{2z} - \frac{1}{6z^3} + \cdots$$

Since this Laurent series has infinitely many negative powers of z, we know $z_0 = 0$ is an essential singularity. The Residue is the coefficient of $1/z$, which we see is $\dfrac{1}{2}$.

Q 9. Compute the following integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{5 + 4\cos\theta}\, d\theta$

Answer This is a perfectly standard trigonometric integral, so we can use the technique of converting it to a contour integral around the unit circle using $z = e^{i\theta}$.

Then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. So the integral becomes

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta = \oint_C \frac{1}{5+2(z+1/z)}\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{dz}{2z^2+5z+2}$$

The integrand has singularities at $z = -2$ and $z = -1/2$; only $z = -1/2$ is relevant since it's inside the unit circle.

Thus

$$\int_0^{2\pi} \frac{1}{5 + 4\cos\theta}\, d\theta = \frac{1}{i} \oint_C \frac{dz}{2z^2 + 5z + 2}$$
$$= 2\pi \operatorname*{Res}_{z=-1/2} \frac{1}{2z^2 + 5z + 2} = 2\pi \frac{1}{4z+5}\Big|_{z=-1/2} = \frac{2\pi}{3}$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+1)}\, dx$

Answer This integral is the real part of the simpler integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+1)}\, dx.$$

To compute this, we use the residues of the complex function at the singularities $z = 2i$ and $z = i$ (since these are the only ones above the real axis).

Thus

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+1)}\, dx$$
$$= 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{iz}}{(z^2+4)(z^2+1)} + 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iz}}{(z^2+4)(z^2+1)}$$
$$= 2\pi i \frac{e^{iz}}{2z(z^2+1)}\Big|_{z=2i} + 2\pi i \frac{e^{iz}}{(z^2+4)(2z)}\Big|_{z=i}$$
$$= 2\pi i \frac{e^{-2}}{(4i)(-3)} + 2\pi i \frac{e^{-1}}{(3)(2i)}$$
$$= \frac{\pi}{3} e^{-1} - \frac{\pi}{6} e^{-2}.$$

This answer is already real, so the desired integral is the same:

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+1)}\, dx = \frac{\pi}{3} e^{-1} - \frac{\pi}{6} e^{-2}$$