Math 241, Spring 2005: Review Sheet for Second Midterm

Be able to:

- Solve the wave equation (not the heat equation) using Laplace transforms. Know how to invert the standard Laplace transforms, including those of the form $e^{-as}F(s)$.
- Compute the Fourier transform, Fourier sine transform, and Fourier cosine transform of a simple function.
- Use the three Fourier transforms to solve the wave, heat, or Laplace equations:
  - If $-\infty < x < \infty$, use the (complex) Fourier transform
  - If $0 \leq x < \infty$ with $u(0,t)$ specified, use the Fourier sine transform
  - If $0 \leq x < \infty$ with $\frac{\partial u}{\partial x}(0,t)$ specified, use the Fourier cosine transform
- Perform basic algebra with complex numbers, including multiplication, division, conjugation, absolute values, arguments, powers, and roots.
- Identify basic sets of complex numbers, and describe how lines get transformed by complex functions.
- Prove that a limit or derivative of a complex function does not exist, by showing that limits horizontally and vertically are different.
- Verify that a function is differentiable (possibly only at certain points) using the Cauchy-Riemann equations.
- Compute the conjugate harmonic function $v(x,y)$, given a harmonic function $u(x,y)$.
- Compute e^z, $\operatorname{Ln} z$, $\ln z$, $\sin z$, $\cos z$, $\tan z$, $\sin^{-1} z$, $\cos^{-1} z$, and $\tan^{-1} z$ for any complex number z.
- Compute a simple contour integral by hand, using a specific parametrization (i.e., without using Cauchy-Goursat or the Cauchy Integral Formula).
- Use the principle of path independence to compute non-closed contour integrals of analytic functions, by finding complex antiderivatives.
- Use the Cauchy Integral Formulas to compute closed contour integrals of analytic functions.
- Determine whether a complex series converges absolutely, converges, or diverges, by applying the n^{th}-term test, the p-series test for $\sum \frac{1}{n^p}$, and the geometric series test for $\sum z^n$.
- Determine the radius of convergence of a complex power series with given coefficients. For simple examples, determine which points on the circle of convergence the power series converges.
- Determine what the radius of convergence must be for a given function's Taylor series at any point, by computing the distance to the nearest singularity.
- Use standard power series (such as for $1/(1-z)$, e^z, $\sin z$, and $\cos z$) to derive the Taylor series of functions like $z/(1-z)^2$, $e^z \cos z$, etc., around any given complex number z_0. This may involve algebra or term-by-term calculus.

Important concepts you should understand:

- The differences between Laplace and Fourier transforms:
  - Laplace transforms are transforms in the time variable and are used only for problems with $0 \le t < \infty$.
  - Fourier transforms are transforms in the space variable and are used only for problems with $0 \le x < \infty$ or $-\infty < x < \infty$. (Thus for example, to solve the Laplace equation on an infinite strip, you transform with respect to the infinite-domain variable only.)
  - Laplace transforms can be used no matter what the domain of x is.
  - Fourier transforms have simple inversion formulas, unlike Laplace transforms.
  - Many equations can be solved using either technique.
  - Laplace transforms give simple answers to the wave equation, difficult answers for the heat equation.
  - Fourier transforms give answers that are of medium complexity to any equation.
  - Neither transform technique changes at all if the equation is nonhomogeneous.
  - Fourier transforms will only work if the solution is assumed to go to zero as the space variable approaches infinity. This condition is often assumed implicitly for the standard PDEs.
- The difference between $\arg z$ and $\operatorname{Arg} z$, as well as between $\ln z$ and $\operatorname{Ln} z$.
- Why rules such as $\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2$ may fail, but $\ln(z_1 z_2) = \ln z_1 + \ln z_2$ are still true.
- Where the Cauchy-Riemann equations come from, and why they imply that the real and imaginary parts of any complex function are harmonic.
- The definition of the complex derivative, and why many complex functions do not have derivatives.
- The difference between being *differentiable* at a point and being *analytic* at the point. For example, if $f(z) = \overline{z}^2$, then f is not analytic at $z = 0$ but $f'(0) = 0$.
- Simple-connectedness vs. multiple-connectedness of a domain. A simply-connected domain is one in which any closed curve can be shrunk down to a point while still being inside the domain. A multiply-connected domain is one for which there is some hole that prevents this shrinking.
- How to compute a contour integral when the contour contains multiple singularities, or when the contour changes orientation.

You should be able to do all of the Core Problems for Sections 15.2–15.4, 17.1–17.8, 18.1–18.4, and 19.1–19.2.

In addition, you should be able to do the following review exercises. *Many of these exercises combine several topics from different sections; these are excellent practice for the exam.*

- Chapter 15: pp. 765–766: 6 (using Laplace transforms); 1, 4, 7, 8, 9, 11, 13 (using Fourier transforms).
- Chapter 17: pp. 824–825: 1–20, 22–24, 26, 30–32, 37–40.
  Note: the review problems here inexplicably do not include much of the later portions of Chapter 17. You may want to practice some of the following problems:
  - (17.5): 1–28
  - (17.6): 23–38
  - (17.7): 1–14 and 21–30
  - (17.8): 1–10
- Chapter 18: pp. 849–850: 1, 3–7, 9–14, 16–25, 27–28.
- Chapter 19: pp. 886–887: 1–2, 11–13.