Math 241, Spring 2005: Review Sheet for Final Exam

Be able to:

- Derive a Laurent series for any rational function, using partial fractions and geometric series.
- Determine which annuli a Laurent series can be constructed in for a given function (by computing the distances from the base point to the isolated singularities).
- Determine that z_0 is a pole of order $n \geq 1$ of $f(z)$, by showing that $\lim_{z \to z_0} (z - z_0)^n f(z)$ exists, is finite, and is not zero.
- Show that z_0 is an essential singularity of $f(z)$ by writing out the full Laurent series (for simple cases).
- Show that z_0 is a removable singularity of $f(z)$, by showing that $\lim_{z \to z_0} f(z)$ exists.
- Compute residues at poles using one of the formulas provided under "Useful formulas." (See below.)
- Evaluate any contour integral using the Residue Theorem. (Note: this replaces the Cauchy Integral Formula and all other techniques of evaluating integrals of analytic functions over closed contours.)
- Be able to evaluate the three types of real integrals we've discussed, using the Residue Theorem:
  - $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, using $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, and $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$.
  - $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$ where $P(x)$ and $Q(x)$ are polynomials with $\deg(Q) \geq \deg(P) + 2$.
  - $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x}\, dx$ where $P(x)$ and $Q(x)$ are polynomials with $\deg(Q) \geq \deg(P) + 1$ and $\alpha > 0$.

Useful formulas:

- For deriving Laurent series, after you've computed the partial fractions:
  - For $|z - z_0| < |a - z_0|$: $\displaystyle\frac{1}{z-a} = -\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(a - z_0)^{n+1}}$
  - For $|z - z_0| > |a - z_0|$: $\displaystyle\frac{1}{z-a} = \sum_{n=1}^{\infty} \frac{(a - z_0)^{n-1}}{(z - z_0)^n}$

  Differentiate term by term as needed for $\dfrac{1}{(z-a)^k}$.

- Residues at simple poles: $\displaystyle\operatorname*{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$, if $g(z_0) \neq 0$.

  This is almost always the fastest method of computing residues; the only time it doesn't work is when you get an infinite value, which means the pole is not simple.

- If $f(z)$ has a pole of order n at z_0, $\displaystyle\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\Big((z - z_0)^n f(z)\Big)$.

  Often you will have to use L'Hôpital's rule to evaluate this limit.

You should be able to do all of the Core Problems for Chapter 19.
In addition, you should be able to do the following review exercises. *Many of these exercises combine several topics from different sections; these are excellent practice for the exam.*

- Chapter 19: pp. 886–887: 3–10, 15–26, 28, 30–34.