Homework for Section 5.4

Mathematics 360

due Monday, December 9

Read *Elementary Classical Analysis*, Section 5.4.

1. Exercises 2–7, pp. 267–268 of *ECA*.

2. The Bessel function J is defined by the power series

$$J(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{x^{2n}}{n!^2}$$

   Prove that J is twice differentiable and that J satisfies the differential equation

$$\frac{d}{dx}\left(x\frac{dJ}{dx}\right) + xJ = 0$$

3. Using the differential equation, show that

$$\frac{d}{dx}\left(J'(x)^2 + J(x)^2\right) = -\frac{2}{x}J'(x)^2$$

   Use this to prove carefully that both J' and J are bounded for $x \geq 0$. What happens for $x \leq 0$?

4. Using an argument similar to the one used in class to study $C(x)$ (the cosine function), show that J has a positive zero. Hint: first show that $xJ'(x)$ is decreasing if $x > 0$ and $J(x) > 0$.

5. Show that J has infinitely many positive zeroes. Hint: Suppose we have already found a zero x_1, with $J(x_1) = 0$ and $J'(x_1) < 0$. Show that J is concave up whenever $x > 0$, $J(x) < 0$, and $J'(x) < 0$. From this, show that J has a local minimum at some point $x > x_1$. Then use the argument from the previous problem to get the next zero. The case where $J'(x_1) > 0$ is very similar.