Homework for Sections 1.4–1.6

Mathematics 360

due Friday, September 27

Read *Elementary Classical Analysis*, pp. 49–63, and the proofs on pp. 86–93.

1. (The contraction principle)

   (a) Suppose that $\{x_n\}$ is a sequence such that, for every $n \in \mathbb{N}$,

   $$|x_{n+2} - x_{n+1}| < r|x_{n+1} - x_n|$$

   for some $r \in]0, 1[$. Prove that $\{x_n\}$ is a Cauchy sequence. (Hint: Example 1.4.8.)

   (b) Give an example of a sequence $\{x_n\}$ satisfying

   $$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$$

   for every $n \in \mathbb{N}$, but such that $\{x_n\}$ is not a Cauchy sequence.

2. Exercise 25, page 99 of *ECA*.

3. Given a sequence x_n, define the *mean sequence* by

   $$m_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

   Show that if $x_n \to x$, then $m_n \to x$. Find an example where m_n converges but x_n does not.

4. Exercises 2 and 5, page 56 of *ECA*. (Prove or find a counterexample.)

5. Construct a sequence with infinitely many distinct cluster points, or prove that this is impossible.

6. Construct a sequence which does not have a limit, but such that every subsequence does have a limit; or prove that this is impossible.

7. Exercise 9, page 98 of *ECA*.

8. Exercise 1, page 63 of *ECA*.