# Homework for Sections 4.6–4.7

Mathematics 360

due Friday, November 8

Read *Elementary Classical Analysis*, Sections 4.6 and 4.7, and all the proofs.

1. *ECA*, pg. 196, exercises 2 and 3.

2. Compute the derivative of $f(x) = \frac{1}{1+x^2}$ both from the definition and using the chain rule, and check that the formulas are equivalent.

3. Prove the geometric interpretation of concavity: that if $f'' > 0$ on an interval $[a, b]$, then $f$ is strictly below its secant line on $]a, b[$. Hint: use proof by contradiction and the mean value theorem.

4. Let $f$ be a function whose second derivative $f''$ exists everywhere in an interval $[x_0 - h, x_0 + h]$ and satisfies $|f''(x)| \leq M$. Let $L$ be the *linearization* of $f$ at $x_0$:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Prove that for $x \in [x_0 - h, x_0 + h]$,

$$|f(x) - L(x)| \leq Mh^2$$

Hint: first show that $|f'(x) - L'(x)| \leq Mh$ using the mean value theorem on $f' - L'$.

5. Prove that if $f$ is differentiable at $x = 0$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, then $f$ is differentiable everywhere and $f'(x) = f'(0)f(x)$ for all $x$.

6. (The false product rule)

   (a) If $f(x) = x$ and $g(x) = 1/(1-x)$, show that $(fg)' = f'g'$ for all $x \neq 1$.

   (b) If $f(x) = x^2$, what does $g$ have to be for $(fg)' = f'g'$ to be true whenever both sides are defined?

7. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at every point (including $x = 0$). However, show that $f$ is not $C^1$; that is, show $f'(x)$ is not continuous everywhere. You may assume standard facts about the sin function, such as continuity, differentiability, and that it is bounded between $-1$ and 1.