Homework for Sections 4.7–4.8

Mathematics 360

due Friday, November 15

Read *Elementary Classical Analysis*, Sections 4.7 (again) and 4.8, and all the proofs.

1. Prove that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
for any n. Use this result to compute
$$\int_0^1 x^2\, dx$$
directly from the definition. (You may take for granted that the function is integrable, since it is continuous.)

2. Define a function $f : [0,1] \to [0,1]$ by
$$f(x) = \begin{cases} \frac{1}{n+2} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } x = 0 \end{cases}$$
Prove that f is integrable on $[0,1]$ and that
$$\int_0^1 f(x)\, dx = \frac{1}{4}$$
Hint: the integral can be written as a telescoping series.

3. Using the chain rule and the fundamental theorem of calculus, prove the substitution formula for integration, assuming that f is continuous and that g' is continuous.
$$\int_a^b f\big(g(x)\big)\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

4. Using the product rule and the fundamental theorem of calculus, prove the formula for integration by parts, assuming that f' and g' are continuous:
$$\int_a^b f(x)g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\, dx$$

5. Exercises 4 and 8, *ECA*, pg. 211.

6. (The mean value theorem for integrals) If f is continuous on $[a,b]$, prove that there is some $c \in]a,b[$ with
$$(b-a)f(c) = \int_a^b f(x)\, dx$$
Show that this property may not be true if f is integrable on $[a,b]$ but not continuous.