# Homework 1: Riemannian metrics

Differential Geometry I

due Tuesday, Sept. 16

1. Problems 1, 4, 6, and 7 of Chapter 1.

2. (a) Do problem 5 of Chapter 0 (embedding of the projective plane $P^2(\mathbb{R})$ in $\mathbb{R}^4$). Use the embedding

$$F(x, y, z) = (\tfrac{1}{2}(x^2 - y^2), xy, xz, yz), \qquad (x, y, z) = p \in \mathbb{R}^3$$

   instead of the one given in the book.

   (b) If $\mathbb{R}^4$ has the standard Euclidean metric, compute the metric induced by this embedding on the projective plane $P^2(\mathbb{R})$, using spherical coordinates. Show that this is not the same metric as in Chap. 1, problem 1.

3. Compute the circumference of a circle $x^2 + (y - a)^2 = b^2$, where $0 < b < a$, in the Lobatchevski metric from Chap. 1, problem 4.

4. Consider the Heisenberg group $H$, the Lie group under multiplication of matrices

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, (x, y, z) \in \mathbb{R}^3 \right\}.$$

   (a) Compute the left-invariant vector fields $E_1$, $E_2$, and $E_3$ which are, at the identity, $E_1 = \partial_x$, $E_2 = \partial_y$, $E_3 = \partial_z$.

   (b) Compute the right-invariant vector fields $F_1$, $F_2$, and $F_3$ which are, at the identity, $F_1 = \partial_x$, $F_2 = \partial_y$, $F_3 = \partial_z$. Express the $F$'s in terms of the $E$'s.

   (c) Is there a bi-invariant metric on $H$?

   (d) Is there a bi-invariant 3-form on $H$?