Homework 3: Geodesics

Differential Geometry I

due Thursday, Oct. 2

1. Prove that if $\{E_1, \ldots, E_n\}$ is an orthonormal basis in some open set $U \subset M$, then the divergence of a vector field X is given by the formula

$$\operatorname{div} X = \sum_{k=1}^{n} \langle E_k, [E_k, X] \rangle.$$

2. (a) Show that on a Lie group with left-invariant metric, γ is a geodesic if and only if the vector $X(t) = L_{\gamma(t)^{-1}}{}_* \frac{d\gamma}{dt}$ satisfies the Euler equation

$$\frac{dX^k}{dt} + c_{kij} X^i(t) X^j(t) = 0,$$

where $c_{pqr} = \langle [E_p, E_q], E_r \rangle$.

   (b) Prove that if the metric is actually bi-invariant, then $X(t)$ is constant. Explain why this means that any 1-parameter subgroup of G is a geodesic. (See do Carmo, Chapter 3, problem 3(a) for a definition and hints.)

   (c) Compute the structure constants of the Heisenberg group with left-invariant metric given by

$$\langle E_i, E_j \rangle = \delta_{ij}.$$

   Use these to find the general solution of the Euler equation on the Heisenberg group.

   (d) Solve the flow equation

$$\frac{d\gamma}{dt} = L_{\gamma(t)_*} X(t), \quad \gamma(0) = e,$$

where $X(t)$ is a solution of the Euler equation with $X(0) = X_0$, to find the geodesics on the Heisenberg group which start at the identity. Use this to obtain an explicit formula for the Riemannian exponential map on the Heisenberg group.

(e) Solve the flow equation

$$\frac{d\gamma}{dt} = L_{\gamma(t)_*}X_0, \quad \gamma(0) = e,$$

to find the Lie group exponential map on the Heisenberg group.

3. Write down the geodesic equation on the upper half plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

for a geodesic $\gamma(t) = (x(t), y(t))$.

(a) Show that γ satisfies the equations

$$\begin{aligned} \dot{x} &= Ly^2 \\ \dot{y}^2 &= y^2 - L^2y^4 \end{aligned}$$

(b) Solve these equations, and show that every geodesic lies on a circle

$$(x - c)^2 + y^2 = \frac{1}{L^2}$$

for some constant c.

(c) Determine the geodesic circles with this metric, i.e. determine the set

$$S_r(0, y_0) = \left\{\exp_{(0,y_0)}(rv_0) \,\middle|\, \langle v_0, v_0\rangle = 1\right\}.$$

Show that geodesic circles are actually Euclidean circles, though not centered at $(0, y_0)$.

(d) Use your formula for the length of a circle in the first homework set to determine what the circumference of a circle is in terms of the radius. How does this compare to the Euclidean formula?