Homework 4: Curvature

Differential Geometry I

due Thursday, Oct. 19

1. Find a formula for the function G so that the metric
$$ds^2 = G^2(x,y)\left(dx^2 + dy^2\right)$$
has constant sectional curvature K.

2. (Counting) The number of "independent" components of a tensor is the minimum number of quantities needed to specify it completely. For example, the metric tensor g_{ij} has n^2 components, but because of the symmetry $g_{ij} = g_{ji}$, only $n(n+1)/2$ of these are independent. Based on Proposition 2.5 of do Carmo, Chapter 4, show that there is only one independent component of the Riemann curvature tensor in dimension $n = 2$, given by R_{1212}. (In other words, show that all other components are either zero or can be determined in terms of this one.) Show that there are six independent components in dimension $n = 3$. What are they?

3. Compute the six independent components of the Riemann curvature tensor for the "warped product" metric
$$ds^2 = dr^2 + \varphi^2(r)\,d\theta^2 + \psi^2(r)\,d\xi^2.$$

4. (a) Compute the sectional curvature $\langle R(E_i, E_j)E_i, E_j\rangle$ on a Lie group with a left-invariant metric, in terms of the structure constants $c_{ijk} = \langle [E_i, E_j], E_k\rangle$ of left-invariant orthonormal vector fields $\{E_1, \ldots, E_n\}$.
   (b) Show how your formula in part (a) simplifies under the assumption that the metric is bi-invariant. (See Chapter 4, problem 1 of do Carmo.)

5. Chapter 4, problem 6.