

Homework 4: More Curvature

Differential Geometry I

due Thursday, Oct. 30

1. On a three-dimensional Riemannian manifold, show how to compute the six independent components of the Riemannian curvature tensor from the six components of the Ricci tensor. (Hint: perform the computation at a point and use Riemannian normal coordinates.)

2. (Computational simplifications.)

Suppose that the metric is, in some coordinates, given by $g_{ij} = \delta_{ij} e^{2\phi_i}$ for some functions ϕ_i , $i = 1, \dots, n$.

- (a) Show that the Christoffel symbols satisfy

$$\Lambda_{ij}^k = \delta_{kj} \partial_i \phi_j + \delta_{ki} \partial_j \phi_i - \delta_{ij} e^{2\phi_i - 2\phi_k} \partial_k \phi_i.$$

- (b) Show that the components of the Ricci curvature tensor

$$R_{ij} = \sum_{k=1}^n \left(\partial_k \Lambda_{ij}^k - \partial_i \Lambda_{kj}^k + \sum_{l=1}^n (\Lambda_{ij}^l \Lambda_{kl}^k - \Lambda_{kj}^l \Lambda_{il}^k) \right)$$

are given by the formula

$$\begin{aligned} R_{ij} = & \partial_i \partial_j (\phi_i + \phi_j - \sum_{k=1}^n \phi_k) + \partial_i \phi_j \partial_j (\sum_{k=1}^n \phi_k) \\ & + \partial_j \phi_i \partial_i (\sum_{k=1}^n \phi_k) - 2 \partial_i \phi_j \partial_j \phi_i - \sum_{k=1}^n \partial_j \phi_k \partial_i \phi_k \\ & + \delta_{ij} \sum_{k=1}^n e^{2\phi_i - 2\phi_k} \left[-2 \partial_k \phi_i (\partial_k \phi_i - \partial_k \phi_k) \right. \\ & \left. - \partial_k \partial_k \phi_i - \partial_k \phi_i \partial_k (\sum_{l=1}^n \phi_l) + 2 (\partial_k \phi_i)^2 \right] \end{aligned}$$

3. Consider a metric on \mathbb{R}^4 of the form

$$ds^2 = \varphi^2(r) dr^2 + \psi^2(r) d\rho^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Show that the Ricci tensor is identically zero iff $\varphi^2(r) = (1+C/r)^{-1}$ and $\psi^2(r) = (1+C/r)$, for some constant C (this is the Schwarzschild metric appearing in general relativity). Show that if $C \neq 0$, the Riemann curvature tensor is nonzero by finding a nonzero sectional curvature.

4. Chapter 4, Exercises 7, 8, and 10.
5. Show that on the Heisenberg group, with an orthonormal basis of left-invariant vector fields satisfying $[E_1, E_2] = 0$, $[E_1, E_3] = E_2$, and $[E_2, E_3] = 0$, the fields E_1 , E_2 , and E_3 are eigenvectors of the Ricci curvature operator and that the scalar curvature is negative.