Homework 6: Jacobi Fields

Differential Geometry I

due Thursday, Nov. 6

1. Chapter 5, problems 3, 4, and 5.

2. On a Lie group with left-invariant metric, the Jacobi equation may be written as

$$\left(\frac{d}{dt} + \nabla_{X(t)}\right)^2 Y(t) + R\big(X(t), Y(t)\big)X(t) = 0,$$

where $X(t)$ is a solution of the Euler equation $\frac{dX}{dt} + \nabla_{X(t)}X(t) = 0$. Show that:

   (a) This equation is equivalent to

$$\frac{d^2Y}{dt^2} + 2\nabla_{X(t)}\frac{dY}{dt} + [Y(t), \nabla_{X(t)}X(t)] + \nabla_{X(t)}[X(t), Y(t)] + \nabla_{[X(t),Y(t)]}X(t) = 0.$$

   (b) If the metric on the Lie group is bi-invariant, then the Jacobi equation can be integrated to yield

$$\frac{dY}{dt} + [X(t), Y(t)] = Z_0$$

   for some vector Z_0.

   (c) If the Lie group is three-dimensional, with a basis E_1, E_2, E_3 of the Lie algebra satisfying

$$[E_1, E_2] = \lambda_3 E_3, \qquad [E_2, E_3] = \lambda_1 E_1, \qquad [E_3, E_1] = \lambda_2 E_2,$$

   then $X(t) = E_1$ is a solution of the Euler equation. Write down the three components of the Jacobi equation.

   (d) If $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$, then $K(E_1, E_2) = -2$ and $K(E_1, E_3) = 2$, and all Jacobi fields are bounded in time. Find the conjugate points.

(e) If $\lambda_1 = 6$, $\lambda_2 = -1$, and $\lambda_3 = 1$, then $K(E_1, E_2) = 14$ and $K(E_1, E_2) = 2$. Almost all Jacobi fields grow exponentially in time. Find the conjugate points.