Math 240 HW 2 Solution

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7.2.2 (a) $\vec{X}(s,t) = (s+t,\ s-t,\ st)$

$\vec{T}_s(s,t) = (1,\ 1,\ t)$, $\vec{T}_t(s,t) = (1,\ -1,\ s)$

$$\vec{T}_s \times \vec{T}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & t \\ 1 & -1 & s \end{vmatrix} = (s+t)\vec{i} + (t-s)\vec{j} - 2\vec{k}$$

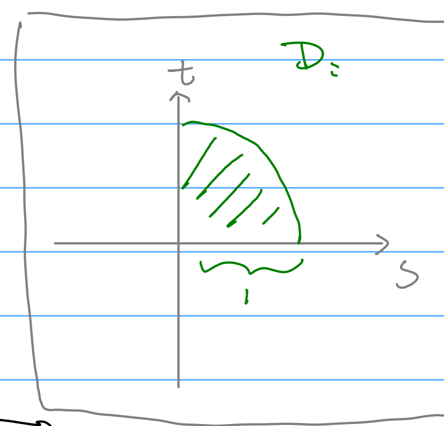
$$\iint_X f\,dS = 4\iint_X dS = 4\iint_D \|\vec{T}_s \times \vec{T}_t\|\,ds\,dt$$

$$= 4\iint_D \sqrt{(s+t)^2 + (t-s)^2 + 4}\ \ ds\,dt$$

$$= 4\iint_D \sqrt{2(s^2+t^2)+4}\ \ ds\,dt$$

(use $s = r\cos\theta$, $t = r\sin\theta$)

$$= 4\int_0^{\pi/2}\int_0^1 \sqrt{2r^2+4}\ \ r\,dr\,d\theta$$



$$= 2\pi \int_0^1 \sqrt{2r^2+4}\ \ r\,dr$$

(Let $\rho = \sqrt{2r^2+4}$, $\rho^2 = 2r^2+4$, $\rho\,d\rho = 2r\,dr$)

$$= \pi \int_2^{\sqrt{6}} \rho \cdot \rho\,d\rho$$

$$= \frac{\pi}{3}\ \rho^3 \Big|_2^{\sqrt{6}} = \boxed{\frac{\pi}{3}(6\sqrt{6} - 8)}$$

(b) Since $\vec{N} = (s+t)\vec{i} + (t-s)\vec{j} - 2\vec{k}$, $\vec{F} = (s+t)\vec{i} + (s-t)\vec{j} + st\,\vec{k}$

$$\iint_X \vec{F} \cdot d\vec{S} = \iint_D (\vec{F} \cdot \vec{N})\ ds\,dt$$

$$= \iint_D \left[(s+t)^2 - (s-t)^2 - 2st\right] ds\,dt$$

$$= \iint_D 2st \, ds\,dt$$

$$= \int_0^{\pi/2} \int_0^1 2r^2 \cos\theta \sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin 2\theta \, d\theta \int_0^1 r^3 \, dr$$

$$= \frac{1}{2}(-\cos 2\theta)\Big|_0^{\pi/2} \cdot \frac{r^3}{4}\Big|_0^1 \quad \boxed{= \frac{1}{4}}$$

7.2.8 (a) $\iint_S x \, dS = 0$ because the sphere is symmetric with respect to $x \Leftrightarrow -x$ operation, i.e., $(-x)^2 + y^2 + z^2 = a^2$ is the same as $x^2 + y^2 + z^2 = a^2$.

To show that, let $\tilde{x} = -x$. Due to the symmetry between x & $\tilde{x}$, we have

$$\iint_S \tilde{x} \, dS = \iint x \, dS$$

Since $\tilde{x} = -x$,

$$-\iint x \, dS = \iint x \, dS.$$

Therefore $\iint x \, dS = 0$

(b) $\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n}) \, dS$

It is well known that $\hat{n} = \hat{r} = \dfrac{x\,i + y\,j + z\,k}{\| r \|}$ on a sphere.

Since now $\| r \| = a$, $\vec{F} = i + j + k$

$$\vec{F} \cdot \hat{n} = \frac{1}{a}(x + y + z).$$

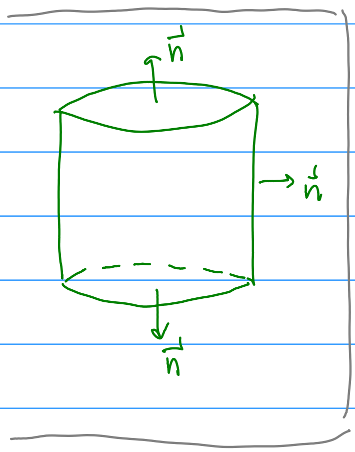
For the similar reason why $\iint_S x \, dS = 0$, we have

$$\iint_S y \, dS = \iint_S z \, dS = 0$$

Therefore $\iint_S \vec{F} \cdot d\vec{S} = 0$

7.2.12 $\iint_S xyz \, dS = \underbrace{\cancel{\iint_{\text{bottom}} xyz \, dS}}_{=0} + \iint_{\text{top}} xyz \, dS + \iint_{\text{lateral}} xyz \, dS$

because $z=0$ at bottom



$$= \underset{\substack{z=4 \\ \text{at top}}}{4} \iint_{\text{disk}} xy \, dS + \int_0^{2\pi}\int_0^4 \underbrace{9\cos\theta\sin\theta}_{\substack{x=3\cos\theta \\ y=3\sin\theta}} \cdot z \underbrace{dz \, 3 \, d\theta}_{dS}$$

$$= 4\int_0^{2\pi}\int_0^3 r^2 \cos\theta \sin\theta \, r \, dr \, d\theta + 27\int_0^{2\pi}\int_0^4 \cos\theta \sin\theta \, z \, dz \, d\theta$$

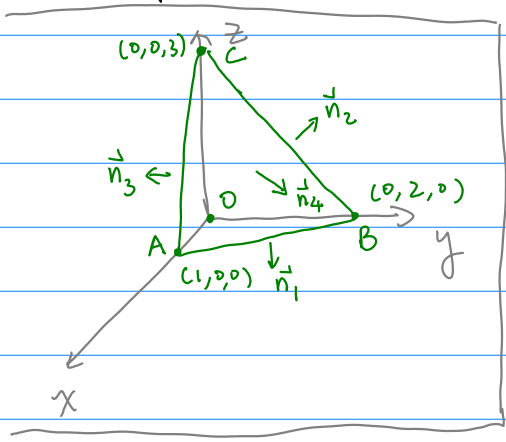
$$= \left(4 \cdot \frac{r^4}{4}\Big|_0^3 + 27 \cdot \frac{z^2}{2}\Big|_0^4\right) \underbrace{\int_0^{2\pi} \frac{1}{2}\sin 2\theta \, d\theta}_{=0}$$

$$\boxed{= 0.}$$

Rmk: We didn't have to calculate anything to get the result. In fact, $\iint_S xyz \, dS = 0$ by symmetry. Since the cylinder is symmetric when $x \leftrightarrow -x$, using the same logic as in 8(a), we immediately get

$$\iint_S xyz \, dS = 0.$$

7.2.26 Since S is a tetrahedron, we need to study four surfaces, all of which triangles.



$$\vec{F} = x^2\vec{i} + 4z\vec{j} + (y-x)\vec{k}.$$

Flux of $\vec{F}$ across S

$$= \iint_S \vec{F} \cdot d\vec{S}$$

$$= \left(\iint_{OAB} + \iint_{OBC} + \iint_{OAC} + \iint_{ABC} \right) \vec{F} \cdot d\vec{S}$$

meaning the triangle OAB

For triangle OAB, it's on $x-y$ plane, therefore we can simply use (x,y) as parameters. The equation for line AB is easy to get:

$$x + \frac{y}{2} = 1 \quad \text{or} \quad x = 1 - y/2$$

Since the normal is outwards, it's $\underline{\underline{-\vec{k}}}$. (see picture)

Similarly for OBC, we use (y,z) and the equation for BC is

$$\frac{y}{2} + \frac{z}{3} = 1 \quad \text{or} \quad y = 2 - \frac{2}{3}z$$

And the normal is $\underline{\underline{-\vec{i}}}$

Similarly for OAC, we use (z,x), with CA's equation

$$x + \frac{z}{3} = 1 \quad \text{or} \quad x = 1 - z/3$$

And the normal $\underline{\underline{-\vec{j}}}$.

Finally for ABC, it's easy to get the equation for ABC as

$$x + \frac{y}{2} + \frac{z}{3} = 1$$

Thinking of (x,y) as parameters, we have

$$z = 3 - 3x - \frac{3}{2}y$$

Therefore $\partial_x z(x,y) = -3, \quad \partial_y z(x,y) = -\frac{3}{2}$.

The normal
$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \end{vmatrix} = 3i + \frac{3}{2} j + k$$

check if pointing outwards. (yes)

Collecting the contributions from all the faces, we have

$$\begin{aligned}
\text{Flux} &= \int_0^2 \int_0^{1-y/2} (-F_z)\, dx\, dy && \leftarrow \text{OAB} \\
&+ \int_0^3 \int_0^{2-\frac{2}{3}z} (-F_x)\, dy\, dz && \leftarrow \text{OBC} \\
&+ \int_0^3 \int_0^{1-z/3} (-F_y)\, dx\, dz && \leftarrow \text{OAC} \\
&+ \int_0^2 \int_0^{1-y/2} \left(3F_x + \frac{3}{2}F_y + F_z\right) dx\, dy && \leftarrow \text{ABC}
\end{aligned}$$

$$= \int_0^2 \int_0^{1-y/2} \underline{(x-y)}\, dx\, dy + \cancel{\int_0^3 \int_0^{2-\frac{2}{3}z} (-x^2)\, dy\, dz} \quad = 0 \text{ since } x=0 \text{ on OBC}$$
$$+ \int_0^3 \int_0^{1-z/3} (-4z)\, dx\, dz + \int_0^2 \int_0^{1-y/2} \left(3x^2 + 6(3-3x-\tfrac{3}{2}y) + \underline{y-x}\right) dx\, dy$$

$$= -4\int_0^3 \int_0^{1-z/3} z\, dx\, dz + \int_0^2 \int_0^{1-y/2} (3x^2 - 18x - 9y + 18)\, dx\, dy$$

$$= -4 \int_0^3 z\left(1-\frac{z}{3}\right) dz + \int_0^2 \left[\left(1-\frac{y}{2}\right)^3 - 9\left(1-\frac{y}{2}\right)^2 - 9\left(1-\underline{y/2}\right)y + 18\left(1-\frac{y}{2}\right)\right] dy$$

let $t = 1 - y/2$

$$= -4\left(\frac{9}{2} - \frac{27}{9}\right) + 2\int_0^1 (t^3 + 9t^2)\, dt$$

$$= -18 + 12 + 2\left(\frac{1}{4} + \frac{9}{3}\right) \boxed{= \frac{1}{2}}$$

Rmk: Instead of doing tedious integration, we could have used the Gauss' Theorem and conclude

$$\text{Flux} = \iiint_V (\nabla \cdot F)\, dV = \iiint_V 2x\, dV$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} 2x \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{2-2x} 2x \left(3-3x-\frac{3}{2}y\right) dy \, dx$$

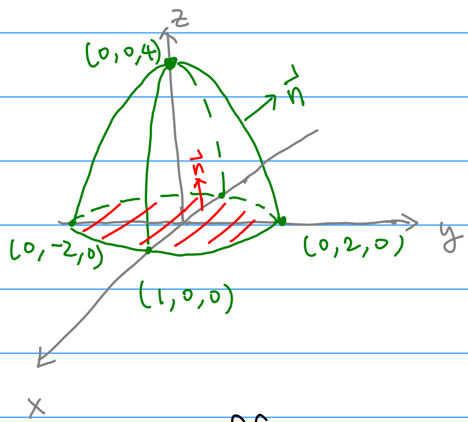
$$= \int_0^1 2x \left[3(1-x)(2-2x) - \frac{3}{4}(2-2x)^2 \right] dx$$

$$= \int_0^1 6x(1-x)^2 \, dx = 6 \int_0^1 (t^2 - t^3) \, dt$$

$$= 6 \cdot \left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{1}{2}}$$

which matches with our previous answer. This way is clearly much shorter.

7.3.12



First let's call the ellipse (the ///// area in the picture) E, assuming it's normal is pointing upwards.

Using Stokes Thm we have

$$\iint_{S-E} (\nabla \times F) \cdot d\vec{S} = 0 \leftarrow S-E \text{ has no boundary}$$

minus b/c orientation

Therefore $\iint_S (\nabla \times F) \cdot d\vec{S} - \iint_E (\nabla \times F) \cdot d\vec{S} = 0$

Since $\nabla \times \vec{F} = zxe^{xy}\,\vec{i} - zye^{xy}\,\vec{j}$

$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_E (\nabla \times \vec{F}) \cdot d\vec{S} = 0$ since $z=0$ on E.

7.3.20 Call the $z=0$ face of the unit cube $\tilde{S}$, normal pointing downwards. Assuming the other faces have also the outwards pointing normals. Using Green's Thm we have

$$\iiint_{[0,1]^3} (\nabla \cdot \vec{F})\, dV = \iint_{S+\tilde{S}} \vec{F} \cdot d\vec{S}$$

Since $\vec{F} = z e^{x^2} \vec{i} + 3y \vec{j} + (z - yz^7) k$

$$\nabla \cdot \vec{F} = 2zx e^{x^2} + 3 - 7yz^6$$

$$\text{LHS} = \int_0^1 \int_0^1 \int_0^1 (2zx e^{x^2} + 3 - 7yz^6)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 [z(e-1) + 3 - 7yz^6]\, dy\, dz$$

$$= \int_0^1 [z(e-1) + 3 - \tfrac{7}{2} z^6]\, dz$$

$$= \frac{e-1}{2} + 3 - \frac{1}{2} = \frac{1}{2}(e+4)$$

On the other hand

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 (\underbrace{yz^7}_{\substack{= 0 \\ \text{since } z=0}} - 2)\, dx\, dy = -2$$

Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_{[0,1]^3} (\nabla \cdot \vec{F})\, dV - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} \boxed{= \frac{1}{2}(e+8)}$$

7.3.24 (a). $\iiint_D (\nabla \cdot F)\, dV = \iint_{\partial D} \vec{F} \cdot d\vec{S}$

∂D has two pieces: the spheres S_7 & S_5.

The induced orientation for S_7 is outwards, & S_5 inwards

Therefore

$$\iiint_D (\nabla \cdot F)\, dV = \oiint_{S_7} \vec{F} \cdot d\vec{S} - \oiint_{S_5} \vec{F} \cdot d\vec{S} = (a \cdot 7 + b) - (a \cdot 5 + b) \boxed{= 2a}$$

(b) If $\vec{F} = \nabla \times \vec{G}$ then

$$\iint_{S_r} \vec{F} \cdot d\vec{S} = \iint_{S_r} (\nabla \times \vec{G}) \cdot d\vec{S} \underset{\text{Thm}}{\overset{\text{Stoke's}}{=}} \oint_{\partial S_r} \vec{G} \cdot d\vec{s} = 0$$

∂S_r is empty

Therefore $ar + b = 0$ for any r

we have $a = 0$ and $b = 0$.