## 4.5

**T/F.2.** True. The number of vectors is larger than the size of the vectors.

**T/F.4.** True. Linear dependence in a subset would indicate that of the whole set.

**Prob.8.** We want to solve

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}.$$

We first do the row operations

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow{A_{12}(1), A_{13}(-2), A_{14}(-3)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & -7 & 4 \end{bmatrix} \xrightarrow{A_{23}(3), A_{24}(7)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

This matrix has no free variable columns, which means the only solution we have is $a = b = c = 0$. Therefore the set of vectors is linearly independent.

**Prob.10.** We want to solve

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}.$$

We first do the row operations

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{A_{12}(-1), A_{13}(-3)} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{A_{23}(2), M_2(-1/3)} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \xrightarrow{A_{21}(-4)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore we have $a = c$ and $b = -2c$. This means $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent and the space spanned by them is described by (taking $c = 1$)

$$x - 2y + z = 0.$$

**Prob.20.** In the basis of $\{1, x\}$, $p_1 = (a, b)$ and $p_2 = (c, d)$. $p_1, p_2$ are linearly independent if and only if

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$$

which is the same as

$$ad - bc \neq 0.$$

**Prob.30.** From $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \tan x$, we get

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\sin x & -\cos x & 2\sec^2 x \tan x \end{vmatrix}$$
$$= \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ 0 & 0 & (2\sec^2 x - 1)\tan x \end{vmatrix} = \frac{\tan x(2 - \cos^2 x)}{\cos^2 x}.$$

When $x \in (-\pi/2, \pi/2)$, $\tan x \neq 0, 0 < \cos x < 1$, therefore $W[f_1, f_2, f_3](x) \neq 0$, and $f_{1,2,3}$ are linearly independent.

## 4.6

**T/F.2.** False. $W$ is isomorphic to any $m$-dimensional subspace of $V$ but doesn't have to be a subspace itself.

**T/F.8.** True. Since 10 is bigger than the dimension of $M_3(\mathbb{R})$ which is 9.

**T/F.10.** True. We can start from any vector in the set and keep adding the linearly independent ones. Since the set spans $V$, this process will end in getting a basis of $V$.

**Prob.2.** We check if $\det([v_1, v_2, v_3]) = 0$ as follows.

$$\det([v_1, v_2, v_3]) = \det \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & -1 \\ 0 & -1 & -2 \end{bmatrix}. = 14 - 1 = 13 \neq 0.$$

Therefore the three vectors given form a basis.

**Prob.6.** We want solve when $\det([v_1, v_2, v_3, v_4]) \neq 0$, which is equivalent to $\{v_1, \cdots, v_4\}$ forming a basis of $\mathbb{R}^4$.

$$\det([v_1, v_2, v_3, v_4]) = \det \begin{bmatrix} 0 & 1 & 0 & k \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ k & 0 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 0 & 1 & 1 & 2 \\ 0 & 0 & k & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 2-k \\ 0 & 0 & k & 1 \end{bmatrix} = 1 - k(2-k) = (k-1)^2.$$

Therefore the set of $k$ that makes the set of vectors given a basis is

$$\left\{ k \in \mathbb{R} \,\middle|\, k \neq 1 \right\}.$$

**Prob.16.** Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. The defining equation of $S$ is then

$$m_{11} + m_{22} = 0.$$

Therefore we can take $m_{12}, m_{21}, m_{22}$ as free variable and have basis as

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

And $\dim(S) = 3$.

**Prob.24.** Since $\dim(P_2) = 3$ we only have to show $p_1, p_2, p_3$ are linearly independent. To that end we use Wronskian.

$$W[p_1,p_2,p_3](x) = \begin{vmatrix} 1+x & x^2-x & 1+2x^2 \\ 1 & 2x-1 & 4x \\ 0 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 2x-1 & 4x \\ 0 & 1 & 2 \\ 1+x & x^2-x & 1+2x^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1+x & -x & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -2 \neq 0.$$

Therefore $\{p_1, p_2, p_3\}$ form a basis of $P_2$.

## 4.8

**T/F.6.** True. The row-echelon form of an $n \times n$ invertible matrix has $n$ leading ones, which correspond to the basis of colspace with $n$ vectors. Therefore $\dim(A) = n$.

**Prob.2.** We do row operations to $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 3 & 4 & -11 & 7 \end{bmatrix}$ as follows.

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 3 & 4 & -11 & 7 \end{bmatrix} \xrightarrow{A_{12}(-3)} \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{A_{21}(-1)} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

Therefore colspace($A$) is simply $\mathbb{R}^2$ and rowspace($A$) is spanned by $\{(1,0,-1,1),(0,1,-2,1)\}$.

**Prob.4.** As above we do row operations to $A = \begin{bmatrix} 0 & 3 & 1 \\ 0 & -6 & -2 \\ 0 & 12 & 4 \end{bmatrix}$ as follows.

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & -6 & -2 \\ 0 & 12 & 4 \end{bmatrix} \xrightarrow{A_{12}(2), A_{13}(-4)} \begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore colspace($A$) spanned by $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and rowspace($A$) is spanned by $\{(0,3,1)\}$.

## 4.9

**T/F.4.** False. Consider the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Its nullity is 1 while all 3 of the diagonal elements are 0.

**T/F.6.** False. A counterexample is when $m \geq n$, we can take any $A$ with nullity 0 and let $B = -A$. Nullity of $A + B$ is $n$ while nullity$(A)$+nullity$(B)$ is 0.

**Prob.10.** We have the linear system with $A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$. We do a series of row operation on $[A\ \mathbf{b}]$ as follows.

$$\begin{bmatrix} 2 & -1 & 1 & 4 & 5 \\ 1 & -1 & 2 & 3 & 6 \\ 1 & -2 & 5 & 5 & 13 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 1 & -1 & 2 & 3 & 6 \\ 2 & -1 & 1 & 4 & 5 \\ 1 & -2 & 5 & 5 & 13 \end{bmatrix} \xrightarrow{A_{12}(-2), A_{13}(-1)} \begin{bmatrix} 1 & -1 & 2 & 3 & 6 \\ 0 & 1 & -3 & -2 & -7 \\ 0 & -1 & 3 & 2 & 7 \end{bmatrix}.$$

$$\xrightarrow{A_{23}(1)} \begin{bmatrix} 1 & -1 & 2 & 3 & 6 \\ 0 & 1 & -3 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{A_{21}(1)} \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -3 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $x_1 = x_3 - x_4 - 1$ and $x_2 = 3x_3 + 2x_4 - 7$. We can write the general solution as

$$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \mathbf{v_p},$$

where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v_p} = \begin{bmatrix} -1 \\ -7 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to check that $\mathbf{v_p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v_1}, \mathbf{v_2}$ are two independent solutions of $A\mathbf{x} = \mathbf{0}$. Since the rank of $A$ is 2 from the row operations, the nullity of $A$ is 4-2=2. Therefore $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis of the null space of $A$.

**Prob.18.** For any $\mathbf{x}$, since $A$ is invertible

$$AB\mathbf{x} = \mathbf{0} \Leftrightarrow B\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Therefore

$$\left\{ \mathbf{x} \,\middle|\, AB\mathbf{x} = 0 \right\} = \left\{ \mathbf{x} \,\middle|\, B\mathbf{x} = 0 \right\},$$

which means their dimensions are the same as well. Therefore

$$\text{nullity}(AB) = \text{nullity}(B).$$