Math 501 HW 1 Shiying

1.2 Define $f(t) = \alpha(t) \cdot V$ then

$$f'(t) = \alpha'(t) \cdot V + \alpha(t) \cdot V'$$
$$= \alpha'(t) \cdot V \quad \text{Since } V \text{ is a constant vector}$$
$$= 0 \quad \text{Since } \alpha'(t) \text{ is orthogonal to } V$$

Therefore $f(t) = C$ is a constant

At $t = t_0$, $f(t_0) = 0 \Rightarrow C = 0$

$\Rightarrow f(t) = 0 \quad \forall t \in I$

which means $\alpha(t)$ is orthogonal to V for all $t \in I$.

1.3 $|\alpha(t)|$ is constant $\Leftrightarrow |\alpha(t)|^2$ is constant

$\Leftrightarrow \alpha(t) \cdot \alpha(t)$ is constant

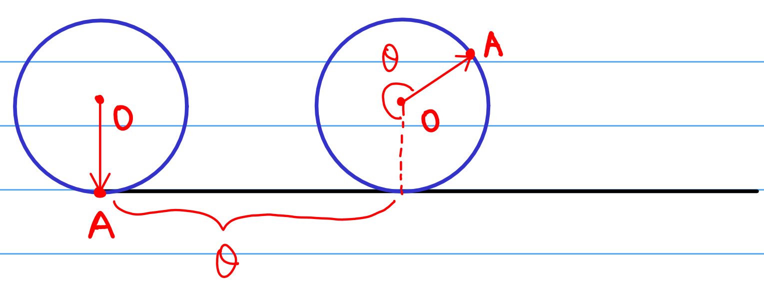
$\Leftrightarrow \left(\alpha(t) \cdot \alpha(t)\right)' = 0 \quad \forall t \in I$

$\Leftrightarrow 2\, \alpha(t) \cdot \alpha'(t) = 0 \quad \forall t \in I$

$\Leftrightarrow \alpha(t)$ is orthogonal to $\alpha'(t) \quad \forall t \in I$

(a)

1.5 For simplicity let's study the pt A, which was at origin before the disk rolls.



Since the rolling has no slipping, at any time

The displacement of O = the rotation of the disk times radius

Use the rotation angle θ as our parameter. The coordinate of O is $(\theta, 1)$

The vector $\vec{OA}$ is $\left(-\cos\left(\theta-\pi/2\right), \sin\left(\theta-\pi/2\right)\right) = \left(-\sin\theta, -\cos\theta\right)$

Therefore the coordinate of A is $(\theta-\sin\theta, 1-\cos\theta)$

This is our desired curve.

(b) $\alpha(\theta) = (\theta-\sin\theta, 1-\cos\theta)$

$\alpha'(\theta) = (1-\cos\theta, \sin\theta)$

$\Rightarrow$ arc length of a complete rotation =

$$\int_0^{2\pi} |\alpha'(\theta)|\, d\theta = \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$= \int_0^{2\pi} \left|2\sin\frac{\theta}{2}\right| d\theta$$

$$= 4\int_0^{\pi} \sin t\, dt = 8.$$

1.6. (a) For any vector V with $|V| = 1$

$$(q-p)\cdot V = V\cdot\int_a^b \alpha'(t)\, dt$$

$$= \int_a^b \alpha'(t)\cdot V\, dt \qquad \text{linearity of integral}$$

$$= \int_a^b |\alpha'(t)|\,|V| \cos\theta(t)\, dt \qquad \theta \text{ is the angle between } \alpha'(t) \text{ \& } V$$

$$\leq \int_a^b |\alpha'(t)|\,|V|\, dt \qquad \cos\theta \leq 1$$

$$= \int_a^b |\alpha'(t)|\, dt \qquad |V| = 1$$

(b) Set $V = \frac{q-p}{|q-p|}$ (assuming $p \neq q$)

$$(q-p)\cdot V = \frac{(q-p)\cdot(q-p)}{|q-p|} = \frac{|q-p|^2}{|q-p|} = |q-p|$$

$$\leq \int_a^b |\alpha'(t)|\, dt$$ as shown in (a)

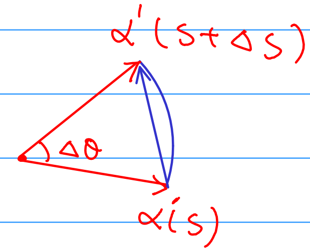
If $q = p$ then $|q-p| = 0$, $\int_a^b |\alpha'(t)|\, dt \geq 0$

In either case we have

$$\underbrace{|q-p|}_{\text{the length of straight line}} \leq \int_a^b \underbrace{|\alpha'(t)|\, dt}_{\text{the length of some arbitrary curve}}$$

Therefore the shortest path connecting any two pts is the straight line.

1.] Let's study s & $s+\Delta s$ two pts. Both $\alpha'(s)$ & $\alpha'(s+\Delta s)$ are unit vectors. Assuming the angle between them is $\Delta\theta$ we have



$$\left|\alpha'(s+\Delta s) - \alpha'(s)\right| = 2\sin\frac{\Delta\theta}{2}$$

Hence $$|\alpha''(s)| = \lim_{\Delta s\to 0}\left|\frac{\alpha'(s+\Delta s)-\alpha'(s)}{\Delta s}\right| = \lim_{\Delta s \to 0} \frac{2\sin\frac{\Delta\theta}{2}}{\Delta s}$$

$$= \frac{d\theta}{ds}$$ = the rate of change of the angle between tangents.

1.8 For a circle with radius r we can write $\tilde{\alpha}(t) = (r\cos t, r\sin t)$

$$\tilde{\alpha}'(t) = (-r\sin t, r\cos t)$$

$\Rightarrow |\tilde{\alpha}'(t)| = r$ which means $ds/dt = r$

Let $s = rt$, $t = s/r$ we have the arclength parameterization of the circle as

$$\alpha(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right)$$

$$\alpha''(s) = \left(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}\right)$$

$$\Rightarrow \kappa(s) = |\alpha''(s)| = \frac{1}{r}.$$