Math 501 HW-2 Shiying

13. For a smooth curve $\alpha : I \to \mathbb{R}^2$. If at t, $\alpha'(t) \neq 0$ then we have a well defined unit tangent vector

$$T = \frac{\alpha'(t)}{|\alpha'(t)|}$$

Since it's in 2-d, given T, we have only two choices for N, corresponding to 2 orientations of T-N frame. Let's pick N to be T rotating $\pi/2$ counterclockwise. To be explicit, if $T = (a, b)$ then $N = (-b, a)$.
We can then define κ as in

$$T'(s) = \kappa(s) N(s).$$

Let the angle of T & x-axis be θ. we have $\kappa(s) = \theta'(s)$.
Of course we could have chosen N to be the opposite direction. In that case κ measures the angular speed of T vector in the clockwise direction. In this homework we'll stick to the counterclockwise one.

14. As we mentioned in 13, $\kappa(s) = \theta'(s)$.
Therefore $\theta(s) = \int_0^s \kappa(\tilde{s})\, d\tilde{s} + \theta_0$ for some θ_0
By definition of $\theta(s)$, $\alpha'(s) = T(s) = \langle \cos\theta(s), \sin\theta(s) \rangle$

$$\Rightarrow \quad \alpha(s) = \left(\int_0^s \cos\theta(\tilde{s})\, d\tilde{s} + a,\ \int_0^s \sin\theta(\tilde{s})\, d\tilde{s} + b \right)$$

for some constant a, b.
The fundamental theorem of calculus shows that the solution is unique up to the initial values a, b, θ_0.
They correspond to the **rigid transformations** of the curve in $\mathbb{R}^2$.

16. (a)

$$r = r(\theta) \Rightarrow \alpha(\theta) = \big(r(\theta)\cos\theta,\ r(\theta)\sin\theta \big)$$

$$\alpha'(\theta) = \big(r'\cos\theta - r\sin\theta,\ r'\sin\theta + r\cos\theta \big)$$

$$= r'(\cos\theta, \sin\theta) + r(-\sin\theta, \cos\theta)$$

Since $(\cos\theta, \sin\theta) \perp (-\sin\theta, \cos\theta)$

$$|\alpha'(\theta)| = (r'^2 + r^2)^{1/2}$$

$$\Rightarrow \quad S = \int_a^b (r'^2 + r^2)^{1/2}\, d\theta$$

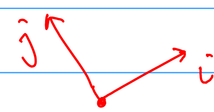
(b) We use the definition that $T'(s) = K(s)\, N(s)$

To simplify notation, let $\vec{i} = (\cos\theta, \sin\theta)$, $\vec{j} = (-\sin\theta, \cos\theta)$

$\vec{i}$ & $\vec{j}$ form an orthonormal frame. In this frame.

$$\alpha'(\theta) = r'\vec{i} + r\vec{j} = |\alpha'(\theta)| \cdot T$$

$$\Rightarrow N = |\alpha'(\theta)|^{-1} (r'\vec{j} - r\vec{i})$$



On the other hand $\vec{i}' = \vec{j}$, $\vec{j}' = -\vec{i}$

$$\Rightarrow \alpha''(\theta) = r''\vec{i} + 2r'\vec{j} - r\vec{i} = (r'' - r)\vec{i} + 2r'\vec{j}$$

Therefore

$$T'(s) = |\alpha'(\theta)|^{-1} T'(\theta) = |\alpha'(\theta)|^{-1} \cdot \left[\frac{\alpha''(\theta)}{|\alpha'(\theta)|} - \frac{\alpha'(\theta)\big(\alpha'(\theta)\cdot\alpha''(\theta)\big)}{|\alpha'(\theta)|^3} \right]$$

$$K(s) = \langle T'(s), N(s) \rangle = |\alpha'(\theta)|^{-3} \langle \alpha''(\theta), r'\vec{j} - r\vec{i} \rangle$$

Since $\langle N, \alpha'(\theta) \rangle = 0$

$$= |\alpha'(\theta)|^{-3} (r^2 - rr'' + 2r'^2)$$

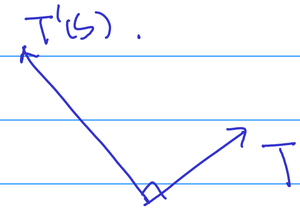
$$= (r^2 - rr'' + 2r'^2) \Big/ (r'^2 + r^2)^{3/2}.$$

17. (a). As in 16 (b).

$$T = \frac{\alpha'(t)}{|\alpha(t)|} \qquad T'(s) = |\alpha'(t)|^{-1} \left[\frac{\alpha''(t)}{|\alpha'(t)|} - \frac{\alpha'(t)\big(\alpha'(t)\cdot\alpha''(t)\big)}{|\alpha'(t)|^3} \right]$$

$$= K(s)\, N$$

Since $T'(s) \perp T$ & $|T| = 1$



$$\kappa(s) = |T'(s)| = |T(s) \times T'(s)|$$

$$= |\alpha'(t)|^{-3} \, |\alpha'(t) \times \alpha''(t)| \qquad \text{since } \alpha'(t) \times \alpha'(t) = 0$$

(b). We use the formula that

$$\tau = \langle N'(s), B \rangle = |\alpha'(t)|^{-1} \langle N'(t), B \rangle$$

Since $B = T \times N = \kappa(s)^{-1} \left(T \times T'(s) \right) = \dfrac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|}$

$$\langle \alpha'(t), B(s) \rangle = \langle \alpha''(t), B(s) \rangle = 0.$$

On the other hand

$$N(t) = \frac{\alpha''(t)\, |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle}{\left| \alpha''(t)\, |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle \right|}$$

The only term in $N'(t)$ that's not linear in $\alpha'(t)$ & $\alpha''(t)$ is

$$\frac{\alpha'''(t)\, |\alpha'(t)|^2}{\left| \alpha''(t)\, |\alpha'(t)|^2 - \alpha'(t) \langle \alpha'(t), \alpha''(t) \rangle \right|} = \frac{\alpha'''(t)}{|\alpha'(t)|^2 \kappa}$$

$$\Rightarrow \tau = |\alpha'(t)|^{-1} \langle N'(t), B \rangle = \frac{\langle \alpha'''(t), \alpha'(t) \times \alpha''(t) \rangle}{|\alpha'(t) \times \alpha''(t)|^2}$$

or in dot notation $\dfrac{\alpha'''(t) \cdot (\alpha'(t) \times \alpha''(t))}{|\alpha'(t) \times \alpha''(t)|^2}$.